

# Recovering Relativity and the Standard Model from the OPH Package Rooted in Observer Consistency

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## Abstract

This paper gives the recovered-core relativity/Standard-Model derivation in Observer-Patch Holography (OPH). Starting from local observer patches on a finite-capacity screen, overlap consistency, and an explicit premise ledger, it reconstructs a schedule-independent overlap normal form, a conditional Lorentz branch, a conditional route to general relativity through a Jacobson-type gravity branch, and a conditional route to the realized Standard Model quotient  $(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))/\mathbb{Z}_6$ . It also derives the exact rational hypercharge lattice and the counting chain  $N_g = 3$ ,  $N_c = 3$  for the realized low-energy matter package. The paper separates theorem-grade fixed-cutoff structure from scaling-limit assumptions, calibration sectors, and phenomenological continuations. The same declared local input  $P$  organizes the electroweak/Higgs trunk and, on the declared exact-release extension surface, the gravity coupling  $G$ .

## 1 Introduction

The standard model of particle physics and general relativity are successful effective descriptions. They leave open a familiar list of questions: why the Lorentz group appears, why the low-energy spacetime description follows a Jacobson-type Einstein branch, why the gauge structure is

$$\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6},$$

why the observed three generations and three colors are realized, why hypercharges take the observed rational values, what fixes the electroweak and flavor scales, and what the framework can and cannot say about the cosmological constant and possible dark-sector continuations.

This paper presents a conditional reconstruction program based on Observer-Patch Holography (OPH). Physical data are organized patchwise: each observer has access only to a patch algebra, neighboring observers must agree on overlaps, and the reconstruction is driven by those consistency conditions. Under the five axioms and the additional branch premises stated below, the same formal structure that produces schedule-independent overlap consistency then feeds a Lorentzian modular-phase classification, a Jacobson-type Einstein branch conditional on fixed-cap stationarity and the null modular bridge together with the bounded-interval projective branch, compact gauge reconstruction in a bosonic internal-gauge sector, charge quantization, and the low-energy architecture selected by the admissibility conditions used in the gauge branch. Any discrete-horizon or black-hole spectroscopy template remains a Phase-III continuation and is not part of the recovered relativity-plus-Standard-Model core.

The paper separates exact structural theorems, conditional scaling-limit branches, calibration-sector outputs, and phenomenological continuations.<sup>1</sup> Several longer arguments are recorded below as proof sketches.

The framework is intended as a candidate fundamental theory in the precise sense relevant for physics: general relativity and the Standard Model appear as effective sectors derived from a smaller axiom set plus an explicit refinement/scaling-limit package. The paper does not use simulation language, observer continuation stories, or other non-falsifiable additions.

## Prioritized achievements

The results emphasized in this paper are ordered below by a combination of impact and defensibility.

1. A conditional Jacobson-type Einstein branch is recovered from observer-overlap modular geometry, an explicit fixed-cap generalized-entropy stationarity premise, the geometric-branch selection burden, and an explicit null-modular scaling branch; in the null modular bridge, quasi-local propagation, endpoint control, half-sided inclusion, the explicit positive null-translation generator, and the half-line generator/charge identification are supplied internally through the derived Borchers stage, while bounded-interval transport through the separate projective branch and the later tensor upgrade remain explicit downstream burdens.
2. The exact Standard Model gauge structure

$$\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}$$

is recovered in the bosonic compact-gauge branch once the MAR admissibility package is imposed.

3. On the realized one-generation chiral matter plus one-Higgs package, anomaly constraints and Yukawa invariance fix the hypercharge lattice, and on the realized MAR-admissible branch the counting chain yields  $N_g = 3$  and then, using Witten's SU(2) anomaly parity constraint,  $N_c = 3$ .
4. The quantitative implementation used here uses two external continuous configuration inputs,  $P$  and  $N_{\text{scr}}$ .
5. Downstream matter-sector continuations, including the charged-lepton centered-readback / centered-to-affine frontier, remain visible but explicitly outside this SM/GR derivation paper's recovered-core theorem package.
6. A separate input-dependent capacity corollary ties the cosmological-constant branch to global screen capacity rather than local vacuum energy.
7. Exclusion of gauge-mediated proton decay follows from product-group structure; any stronger proton-lifetime claim remains UV/EFT/QCD-dependent.
8. Dark-sector, baryogenesis, string, and spectroscopy branches are kept visible only as phenomenological continuations or, where additional ansätze are involved, conjectural phenomenology.

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<sup>1</sup>Broader interpretive questions are deferred here so the claim-tier boundary remains explicit.

9. Overlap repair admits a unique schedule-independent normal form on the stated finite patch-net hypotheses.
10. The fixed-cutoff topological UV package closes on the ordinary branch, the central-defect branch, and the genuinely noncentral branch through a compact crossed-module higher-gauge collar theorem; the remaining UV-side burdens are the continuum BW/geometric lift and the refinement-limit compact-gauge premises.

On that same fixed-cutoff consensus surface, the status split is sharp: normal-form computation is a finite-state decidable problem with the Lyapunov step bound from the consensus paper, the only present approximate-stability inputs are the collar-local splice and record estimates carried there, and no universality theorem for a uniform family of growing patch nets and repair laws is claimed in this paper.

## 2 Overview of Results

This section summarizes the claim tiers used throughout. The labels  $D1$ – $D12$  are internal documentary node labels for the OPH dependency ledger, not bibliography references or external notation. The public paper ledger has no separate  $D11$  row; the other OPH papers should use only the labels defined in the ledger below. The table below is the local key for this SM/GR derivation paper, and the other OPH papers reuse the same labels when they point back to this ledger. The three-phase boundary is

$$\text{Phase I} = (D1\text{--}D5) \cup (D7\text{--}D9), \quad \text{Phase II} = D6 \cup D10, \quad \text{Phase III} = D12.$$

Phase I is the recovered-core claim set. Phase II contains the separate input-dependent capacity corollary and the integrated  $D10$  gauge-calibration sector. Phase III collects phenomenological continuations. The canonical five-axiom basis, the two external continuous inputs, the scaling/continuum premises, and the gauge-side assumptions used below are fixed in Section 3. Each row names a node in the reconstruction program, records its immediate OPH-side parents, separates imported standard mathematics from extra non-axiom physical premises or external inputs, and states the resulting claim tier.

Table 1: Dependency checklist for the OPH SM/GR reconstruction program.

Node	Output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Claim tier
<b>D1</b>	Unique schedule-independent normal form on the physical quotient, together with observable-level confluence on the declared fixed-cutoff physical algebras for the recovery-derived repair relation	overlap-consistency problem on a finite patch net together with the declared fixed-cutoff collar recovery step, its touched-overlap local-fit acceptance contract, support-local disjoint commutation, restriction-compatible union-collar gluing on the physical quotient, and the fixed-point / quotient-local physical observable algebras carried by that same collar package	local-to-global Lyapunov descent on the inconsistency potential; local-diamond completion from overlap-associative center-sector / higher-gauge union-collar gluing; Newman's lemma	repair completeness; stated Petz support/CPTP control where that branch is used	Phase I structural theorem

Table 1: Dependency checklist for the OPH SM/GR reconstruction program.

Node	Output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Claim tier
D2	Collar Markov/entropy split and $G$ -dictionary setup (Theorem 5.2, Propositions 5.11, 5.12, 5.16, Definition 5.17)	Axioms 3.1–3.4	HJPW Markov structure theorem; Petz and Fawzi–Renner recovery theory	exact block-factorized identities require exact Markovity or an explicitly stated idealized limit; the $A/(4G)$ step is the coarse-grained dictionary of Definition 5.17	Phase I structural lemma/proposition layer
D3	Lorentz branch $\text{Conf}^+(S^2) \cong \text{SO}^+(3, 1)$ (Theorem 6.3, Corollary 6.5)	Axioms 3.1–3.4	Bisognano–Wichmann modular geometry; conformal-group classification	T2 together with the theorem-local geometric cap-pair extraction and ordered cut-pair rigidity hypotheses of Theorem 6.3; Theorem 6.2 fixes the target subnet internally	Phase I conditional scaling-limit theorem
D4	Null modular bridge to $T_{kk}$ (Propositions 6.6, 6.9; Corollaries 6.7, 6.11; Theorem 6.10, Lemma 6.12; Theorem 6.13)	D2+D3 and Axioms 3.1–3.4	Borchers–Wiesbrock half-sided modular inclusion; distributional differentiation on half-lines	the theorem-local inherited-strip decomposition and exact-or-controlled Markov hypotheses of Proposition 6.6 and Corollary 6.7, plus T2; quasi-local propagation and endpoint control are supplied internally by Axiom 3.3 on the local finite-constraint branch; the imported continuum input is the standard local modular-Hamiltonian formula for the same half-line family, and bounded-interval formulas still require the separate interval-preserving projective branch	Phase I conditional bridge theorem
D5	Jacobson-type Einstein branch: rest-frame relation plus conditional tensor upgrade (Lemma 6.18, Theorem 6.19, Corollary 6.20)	D3+D4 and Axioms 3.1–3.4	Jacobson small-ball first law; null-to-tensor reconstruction modulo a metric term	T3 fixed-cap stationarity; locally Lorentzian $d = 4$ scaling regime; the separate interval-preserving projective branch used to transport the D4 half-line generator/charge identification to bounded intervals; small-ball constancy assumptions; D4 carried remainder negligible at order $\ell^4$ ; all local directions/reference states for the tensor upgrade	Phase I conditional scaling-limit theorem/corollary
D6	Capacity branch for $\Lambda$ , plus a legacy capacity-level neutrino side estimate (Proposition 6.22, Corollary 6.23)	D5 leaves a $+ \Lambda g_{ab}$ ambiguity	de Sitter entropy relation and dimensional analysis	external input $N_{\text{scr}}$ and the screen-capacity identification	Phase II input-dependent corollary / estimate
D7	Compact gauge reconstruction (Theorem 7.1)	Axioms 3.1–3.4	Doplicher–Roberts / Tannaka reconstruction	Assumption 3.7 on the ordinary or central-defect bosonic branch; the genuinely noncentral fixed-cutoff branch is handled separately by Theorems 5.3–5.6 and is not itself the ordinary compact-group reconstruction theorem	Phase I conditional structural theorem
D8	Product gauge structure up to finite quotient (Lemmas 7.3–7.8, Theorem 7.9)	D7 + Axiom 3.5	compact Lie representation classification; Schur’s lemma	the same ordinary or central-defect bosonic branch as D7, together with a connected positive-dimensional Lie admissible class, one connected abelian factor, and faithful action on the minimal coupled carrier	Phase I realized-branch theorem
D9	Realized Standard Model quotient, hypercharges, the counting chain $N_g = 3$ then $N_c = 3$ , and product-group corollaries (Theorem 7.10, Corollaries 7.11, 7.12, Proposition 7.13)	D8	anomaly-cancellation algebra; Witten global-anomaly argument; CKM CP counting	realized one-generation chiral matter plus one Higgs package; asymptotic-freedom and MAR admissibility premises	Phase I realized-branch theorem/corollary chain

Table 1: Dependency checklist for the OPH SM/GR reconstruction program.

Node	Output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Claim tier
D10	Forward gauge-coupling closure and target-free source-only electroweak readout of the integrated D10 calibration package	D9 + external input $P$	printed RG evolution and matching conventions	pixel constraint and the source-only forward transmutation solve $\mathcal{F}(\alpha_U; P) = 0$ ; the fixed-cutoff edge heat-kernel / Casimir theorem on the microphysics surface together with the compact-group / Peter-Weyl lift used on the D10 lane; printed beta-function and threshold conventions	Phase II calibration sector
D12	Charged-lepton centered-readback / centered-to-affine frontier, texture, dark-sector, baryogenesis, black-hole spectroscopy, proton-spin, proton-lifetime estimates beyond the gauge-channel exclusion, conditional string/worldsheet reorganizations, conjectural critical-superstring extensions, and other continuations	various subsets of D6, D9, and D10	branch-specific EFT and phenomenological manipulations	additional ansätze such as the uniform $\mathbb{Z}_6$ center-label ensemble, discrete texture choices, dark-sector response assumptions, discrete-horizon assumptions, or extra worldsheet/CFT premises	Phase III phenomenological continuation branch

The recovered core of this SM/GR derivation paper is D1–D5 together with D7–D9. D6 is a separate input-dependent corollary, D10 is the integrated calibration sector, and D12 collects phenomenological continuations. The local/global handoff is exactly the  $D5 \rightarrow D6$  handoff: D5 recovers the local Einstein branch only modulo a metric term, and D6 fixes that remaining constant only after the global screen-capacity input  $N_{\text{scr}}$  is supplied. The modular-anomaly dark-sector branch belongs to D12 and does not derive MOND/RAR-style response laws or galaxy/cluster phenomenology. The only external continuous inputs are  $N_{\text{scr}}$  in D6 and  $P$  in D10, together with descendants built from those branches.

### 3 Five Axioms, Two External Continuous Inputs in the Current Quantitative Implementation, and Technical Premises

This section states the premise ledger and dependency map for this SM/GR derivation paper. The OPH basis used below consists of the five core axioms, the two external continuous inputs listed next, the scaling/continuum premises collected afterward, and the gauge-side assumptions of Assumption 3.7. On the fixed-cutoff local finite-constraint MaxEnt branch, the local-Gibbs form, quasi-local dynamics, Lieb–Robinson propagation control, and endpoint-Lipschitz interval control are internal consequences rather than theorem-external regularity selectors. Any Dobrushin/local-Gibbs/mixing language used below names only a sufficient fixed-cutoff collar-recoverability mechanism on that branch. It is not an infinite-volume uniqueness claim for the refinement-limit theory, and it does not by itself constrain whether a transportable refinement-directed edge-sector colimit exists or whether, if it exists, it is trivial or nontrivial. The paper therefore treats any later nontrivial transportable sector colimit as explicit gauge-branch data, not as an entropy-side consequence of the MaxEnt/mixing package. What Axiom 3.3 fixes under refinement is the realized state-side MaxEnt branch itself. Theorem 6.2 fixes the operational/geometric split at fixed cutoff, and Theorem 6.3 fixes the cap generator and its  $2\pi$  normalization once its theorem-local geometric cap-pair extraction and ordered cut-pair rigidity hypotheses are supplied on that geometric subnet. Corollary 6.11

derives the null half-sided modular pair only after the later null blow-up step; and the remaining canonical theorem-external items are therefore the scaling-limit scope clause, the bounded-interval transport/projective branch where invoked, fixed-cap stationarity, and the bosonic ordinary/central gauge premises recorded in Assumption 3.7. The genuinely noncentral fixed-cutoff branch is tracked instead by Theorems 5.3 and 5.6; it is not itself the ordinary compact-group theorem used later.

The formulation uses two external continuous configuration inputs:

$$P \equiv a_{\text{cell}}/\ell_P^2 = 1.63094, \quad (1)$$

$$N_{\text{scr}} \equiv \log \dim \mathcal{H}_{\text{tot}} \sim 10^{122}. \quad (2)$$

In the quantitative implementation used here,  $P$  is treated as the external particle-physics scale input for the D10 calibration sector and  $N_{\text{scr}}$  is inferred from the observed cosmological constant. The first feeds the forward D10 particle-physics map. The second enters the cosmological-capacity branch. Any retained capacity-level neutrino side estimate on that input is legacy bookkeeping rather than the live weighted-cycle theorem lane.

**Axiom 3.1** (Screen Net). *Physical data is organized on a horizon screen  $S^2$  carrying a net of local algebras*

$$P \mapsto \mathcal{A}(P)$$

for connected patches  $P \subset S^2$ , with isotony

$$P \subset Q \implies \mathcal{A}(P) \subset \mathcal{A}(Q).$$

**Axiom 3.2** (Overlap Consistency). *For overlapping patches  $P_1 \cap P_2 \neq \emptyset$ , the local states induced on the shared algebra agree:*

$$\omega_{P_1}|_{\mathcal{A}(P_1 \cap P_2)} = \omega_{P_2}|_{\mathcal{A}(P_1 \cap P_2)}.$$

**Axiom 3.3** (Local MaxEnt and Refinement Stability). *At the regulator scale  $\ell_{\text{UV}}$ , the realized branch is selected by maximizing entropy subject to a finite family of gauge-invariant local constraints*

$$\mathcal{C}_{\ell_{\text{UV}}} = \{O_a(x)\}_{a=1}^{N_{\text{con}}},$$

where each  $O_a(x)$  is supported in a ball of radius  $O(\ell_{\text{UV}})$  and the label set  $a$  is independent of the number of regulator cells. Under refinement, the same finite constraint family is preserved, so the realized states at different cutoffs belong to one common finite-dimensional MaxEnt family. The realized low-energy branch is the refinement-stable branch of that family; symmetry-allowed relevant operators are therefore held at zero on that branch only by symmetry or because they lie in the explicitly retained constraint family. This is a statement about branch persistence, not a universal entropy-ordering theorem for arbitrary phases away from that branch.

**Derived local-Gibbs branch.** On the finite regulator realization of Axiom 3.1, the standard finite-dimensional Lagrange-multiplier argument applied to Axiom 3.3 gives

$$\omega_{\ell_{\text{UV}}}(\lambda) = Z_{\ell_{\text{UV}}}(\lambda)^{-1} \exp(-K_{\ell_{\text{UV}}}(\lambda)),$$

with

$$K_{\ell_{\text{UV}}}(\lambda) = \sum_x \sum_{a=1}^{N_{\text{con}}} \lambda_a O_a(x) + \sum_b \mu_b Q_b,$$

where the  $Q_b$  are the finitely many optional global conserved-charge constraints. Thus the logarithm of the selected state is itself a quasi-local UV generator built from the same bounded-support densities that define the branch.

**Derived quasi-local propagation and interval control.** Fix the regulator graph metric coming from cell adjacency and let  $\tau_t^{\ell_{UV}}$  denote the automorphism group generated by  $K_{\ell_{UV}}(\lambda)$ , or more generally by any branch generator lying in the norm-closed algebra generated by the same bounded-support local densities. Standard Lieb–Robinson estimates for finite-range interactions on the finite regulator net then give constants  $C, \xi, v_{\ell_{UV}} < \infty$  such that for local observables  $A_X, B_Y$ , [2]

$$\|[\tau_t^{\ell_{UV}}(A_X), B_Y]\| \leq C \|A_X\| \|B_Y\| \min(|X|, |Y|) e^{-(d(X,Y) - v_{\ell_{UV}}|t|)/\xi}.$$

So finite-velocity quasi-local propagation is branch-internal: it is the local finite-constraint MaxEnt branch written in propagation form, not an extra selector.

The same locality statement yields the bounded-interval endpoint control used later in the null modular chain. Whenever later sections use a fixed-cutoff interval or collar generator on this branch, it is the corresponding restriction of the same local density list, with the central endpoint term separated off exactly as in the later edge-center bookkeeping. Changing an interval  $I$  to  $I'$  then alters only an  $O(|I'\Delta I|)$  collar of local terms. Hence on any fixed local-energy-bounded domain one has

$$|\langle \psi, (K[I'] - K[I])\phi \rangle| \leq C_{\psi, \phi, I_{\max}} |I'\Delta I|,$$

which is precisely the endpoint-Lipschitz / finite-variation control used later in the null modular chain. This paper does *not* separately derive a second independent microscopic generator beyond this quasi-local branch generator; if a UV completion carries one, the only later requirement is that it lie in the same bounded-support algebraic closure so that the same propagation constants control the selected branch.

This local-Gibbs/Lieb–Robinson package, together with any Dobrushin-type estimate used later, should be read only as fixed-cutoff collar-local recoverability, support control, and carried-error bookkeeping on the selected branch. One may impose such estimates on a chosen finite collar model without deciding the refinement-limit gauge phase. The package is therefore not a proof that the refinement limit lands in a trivial thermodynamic phase, not a proof that transportable edge sectors persist under refinement, not a proof that the relevant sector colimit exists, and not a proof that such a colimit is nontrivial. Its theorem-level role is precisely the collar-local mixing and support/error control carried into the later fixed-cutoff and scaling-limit arguments.

**Internal refinement notion.** Choose any family of refinement channels  $\Phi_{\ell \rightarrow L}$  compatible with Axiom 3.3. Because the same finite local constraint family is preserved, the regulator-scale realized states are parameterized by one common finite-dimensional multiplier space  $\lambda$ , and on the realized branch one may write

$$\Phi_{\ell \rightarrow L}(\omega_{\ell}(\lambda)) = \omega_L(R_{\ell \rightarrow L}(\lambda))$$

for an induced map  $R_{\ell \rightarrow L}$  on that multiplier space. The refinement-stable realized branch is therefore the persistent trajectory or invariant subset selected inside this finite-dimensional space, not an imported theorem-external regularity package. This state-side refinement notion is enough to compare realized states across cutoffs. It does not, by itself, upgrade fixed-cutoff edge labels to transportable refinement-persistent sectors, and it does not show that the directed colimit of such sectors exists or is nontrivial. Any Dobrushin/local-mixing hypothesis used later is logically separate: it supplies a collar estimate on a fixed finite-dimensional model, not information about the refinement-limit gauge phase. Accordingly, whenever later sections speak of a “refinement-stable directed colimit” of edge sectors, the entire transportable directed-system/fiber clause is the explicit content of Assumption 3.7; Axiom 3.3 supplies only the realized state branch along which one asks whether such sector transport persists.

**Axiom 3.4** (Recoverable Generalized Entropy). *A generalized entropy functional exists on caps,*

$$S_{\text{gen}}(C) = S_{\text{bulk}}(C) + \langle L_C \rangle,$$

where  $L_C$  is a positive edge-center entropy functional. In the semiclassical scaling branch, its leading coarse-grained contribution is identified with  $A(\partial C)/(4G)$ . The functional obeys quantum focusing on null generators, and collar tripartitions have small conditional mutual information with controlled recovery maps [11, 12, 13, 14].

**Axiom 3.5** (Minimal Admissible Realization). *Among realized sector packages  $\mathfrak{S}$  consisting of the connected Lie gauge-sector image relevant in the low-energy EFT, its admissible light chiral matter content, and one Higgs doublet, and which are loop-coherent, anomaly-free, refinement-stable with light chiral matter, single-Higgs Yukawa-completable with one connected abelian charge factor acting nontrivially on the coupled carrier, intrinsically CP-capable, and weak-sector UV-completable, the realized package is the lexicographically minimal one under*

$$C(\mathfrak{S}) = (\chi_{\text{cpl}}, N_{\text{nonab}}, N_c, N_g).$$

$\mathfrak{S}$  is the sector package on which MAR acts; it is not the bare tensor category alone. MAR is therefore an explicit structural-economy axiom on admissible realized low-energy branches, not a theorem derived from the preceding axioms.  $\chi_{\text{cpl}}$  is the coupled carrier dimension: the smallest unitary carrier containing a common irreducible block on which the admissible pseudoreal and complex nonabelian charge types both act nontrivially. This is stronger than the abstract minimal faithful representation dimension.

**Assumption 3.6** (Scaling and stationarity premises). *This SM/GR derivation paper uses one canonical numbered list for the external scaling and stationarity premises. The fixed-cutoff realized presentation, the local-Gibbs form, quasi-local Lieb–Robinson propagation, endpoint-Lipschitz interval control, the induced finite-dimensional refinement branch, and the operational/geometric split of Theorem 6.2 are not listed here because they are established internally from Axioms 3.1–3.3. The remaining external items outside the gauge branch are:*

**T2** *all Lorentz/null-modular/Einstein statements are claims about a controlled refinement/scaling limit of the regulator net, not literal exactness at fixed finite cutoff;*

**T3** *fixed-cap generalized-entropy stationarity for the admissible first-variation class used in the Jacobson branch;*

*The bosonic ordinary/central gauge premises are stated separately in Assumption 3.7.*

**Assumption 3.7** (Bosonic gauge reconstruction premises). *Whenever compact gauge reconstruction is invoked on the ordinary or central-defect branch, the following additional premises are kept explicit.*

(G1) *loop coherence on that ordinary or central-defect branch, written  $[z] = 0$ ;*

(G2) *symmetric braiding in the relevant 3 + 1-dimensional EFT branch;*

(G3) *a directed colimit  $\text{Sect}_\infty = \varinjlim_r \text{Sect}_r$  of transportable edge-sector categories with objectwise finite-dimensional fibers;*

(G4) *a faithful bosonic fiber functor on  $\text{Sect}_\infty$ .*

*The genuinely noncentral fixed-cutoff branch is handled separately by Theorems 5.3–5.6 and is not itself the ordinary compact-group reconstruction theorem.*

**Remark 3.8** (Log convention). *Unless an explicit base is written, logarithms and entropies in this paper use natural logs (nats). Base-2 logarithms are written as  $\log_2$  and quoted in bits.*

## Dependency DAG

The table below is the dependency map for this SM/GR derivation paper. The “Immediate OPH ingredients” column records only parents internal to the OPH program. Imported standard mathematics and extra non-axiom physical premises or external inputs are stated separately rather than hidden inside the word “derived.”

Node	Theorem labels and output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Status
<b>D1</b>	Theorem 4.3: unique normal form and schedule independence for the declared recovery-derived repair relation	overlap-consistency problem on a finite patch net together with the declared fixed-cutoff collar recovery step, its touched-overlap local-fit acceptance contract, support-local disjoint commutation, and restriction-compatible union-collar gluing on the physical quotient	local-to-global Lyapunov descent on the inconsistency potential; local-diamond completion from overlap-associative center-sector / higher-gauge union-collar gluing; Newman’s lemma	repair completeness; stated Petz support/CPTP control where that branch is used	structural theorem
<b>D2</b>	Theorem 5.2, Propositions 5.11, 5.12, 5.16, Definition 5.17: collar Markov/entropy layer	Axioms 3.1–3.4	HJPW Markov structure theorem; Petz and Fawzi–Renner recovery theory	exact Markov identities require exact Markovity or an explicitly stated idealized recoverability limit; the $A/(4G)$ identification is the coarse-grained dictionary of Definition 5.17	structural lemma/proposition layer
<b>D3</b>	Theorem 6.3, Corollary 6.5: geometric modular flow and Lorentz branch	Axioms 3.1–3.4	Bisognano–Wichmann modular geometry; conformal classification on $S^2$	T2 together with the theorem-local geometric cap-pair extraction and ordered cut-pair rigidity hypotheses of Theorem 6.3; Theorem 6.2 fixes the target subnet internally	conditional scaling-limit theorem

Node	Theorem labels and output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Status
<b>D4</b>	Propositions 6.6, 6.9; Corollaries 6.7, 6.11; Theorem 6.10, Lemma 6.12; Theorem 6.13: null modular bridge to $T_{kk}$	D2+D3 and Axioms 3.1–3.4	Borchers–Wiesbrock half-sided modular inclusion; distributional differentiation on half-lines	the theorem-local inherited-strip decomposition and exact-or-controlled Markov hypotheses of Proposition 6.6 and Corollary 6.7, plus T2; quasi-local propagation and endpoint control are supplied internally by Axiom 3.3 on the local finite-constraint branch; the imported continuum input is the standard local modular-Hamiltonian formula for the same half-line family, and bounded-interval formulas still require the separate interval-preserving projective branch	conditional bridge theorem
<b>D5</b>	Lemma 6.18, Theorem 6.19, Corollary 6.20: Jacobson-type Einstein branch via a rest-frame relation plus conditional tensor upgrade	D3+D4 and Axioms 3.1–3.4	Jacobson small-ball first law; Lemma 6.14 for the null-to-tensor upgrade modulo a metric term	T3 fixed-cap stationarity; locally Lorentzian $d = 4$ scaling regime; small-ball constancy assumptions; D4 carried remainder negligible at order $\ell^4$ ; all local directions/reference states for the tensor upgrade	conditional scaling-limit theorem/corollary
<b>D6</b>	Proposition 6.22, Corollary 6.23: capacity branch for $\Lambda$ , plus a legacy neutrino side estimate	D5, which leaves the $+\Lambda g_{ab}$ ambiguity	de Sitter entropy relation and dimensional analysis	external input $N_{scr}$ and the screen-capacity identification	input-dependent corollary / estimate
<b>D7</b>	Theorem 7.1: compact gauge reconstruction	Axioms 3.1–3.4	Doplicher–Roberts / Tannaka reconstruction	Assumption 3.7 on the ordinary or central-defect bosonic branch; the genuinely noncentral fixed-cutoff branch is handled separately by Theorems 5.3–5.6 and is not itself the ordinary compact-group reconstruction theorem	conditional structural theorem
<b>D8</b>	Lemmas 7.3–7.8, Theorem 7.9: product gauge structure up to finite quotient	D7 + Axiom 3.5	compact Lie representation classification; Schur’s lemma	the same ordinary or central-defect bosonic branch as D7, together with a connected positive-dimensional Lie admissible class, one connected abelian factor, and faithful action on the minimal coupled carrier	realized-branch theorem
<b>D9</b>	Theorem 7.10, Corollaries 7.11, 7.12, Proposition 7.13: realized Standard Model quotient, hypercharges, the counting chain $N_g = 3$ then $N_c = 3$ , and product-group corollaries	D8	anomaly-cancellation algebra; Witten global-anomaly argument; CKM CP counting	realized one-generation chiral matter plus one Higgs package; asymptotic-freedom and MAR admissibility premises	realized-branch theorem/corollary chain

Node	Theorem labels and output	Immediate OPH ingredients	Standard mathematics used	Extra non-axiom physical premises / external inputs	Status
<b>D10</b>	forward gauge-coupling closure and target-free source-only electroweak readout of Section 7	D9 + external input $P$	printed RG evolution and matching conventions	pixel constraint and the source-only forward transmutation solve $\mathcal{F}(\alpha_U; P) = 0$ ; the fixed-cutoff edge heat-kernel / Casimir theorem on the microphysics surface together with the compact-group / Peter-Weyl lift used on the D10 lane; printed beta-function and threshold conventions	calibration sector
<b>D12</b>	Sections 8–10 continuations: charged leptons, proton spin/lifetime, and string/worldsheet branches	various subsets of D6, D9, and D10	branch-specific EFT and phenomeno-logical manipulations	additional ansätze such as the uniform $\mathbb{Z}_6$ center-label ensemble, texture choices, dark-sector response assumptions, discrete-horizon assumptions, or extra worldsheet/CFT premises	phenomeno-logical continuation

The dependency DAG records the OPH parents, imported mathematics, and extra premises for each node.

**Theorem 3.9** (Summary theorem for the recovered relativity-plus-Standard-Model core). *This theorem packages the recovered-core nodes D1–D5 and D7–D9 of the dependency DAG above. Node D6 is a separate input-dependent corollary, while D10 and D12 remain outside this theorem’s scope. Assume Axioms 3.1–3.4, Axiom 3.5, Assumptions 3.6 and 3.7, and the hypotheses of Theorems 4.3, 6.3, 6.13, 6.19, 7.1, 7.9, and 7.10, together with Corollaries 7.11, 7.12, and Proposition 7.13. Then, in the controlled refinement/scaling regime singled out by Assumption 3.6:*

- (i) *overlap repair admits a unique schedule-independent normal form;*
- (ii) *cap modular flow on the extracted prime geometric cap pair is geometric and yields the connected Lorentz group*

$$\text{Conf}^+(S^2) \cong \text{SO}^+(3, 1);$$

- (iii) *admissible fixed-cap stationarity together with the null modular bridge and the bounded-interval projective branch yield the Jacobson-type rest-frame relation*

$$\delta(G_{00} + \Lambda g_{00}) = 8\pi G \delta\langle T_{00} \rangle$$

*at the cap center in the diamond rest frame; if that rest-frame relation holds for all local directions and reference states in the scaling regime, then the null-to-tensor upgrade gives*

$$G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab} \rangle;$$

- (iv) *under Assumption 3.7, the edge-sector category reconstructs a compact gauge group, and the realized connected gauge structure has the form*

$$\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\Gamma}$$

*for some finite central subgroup  $\Gamma$ .*

After the hypercharge lattice, generation-count, color-count, and trivial-action quotient steps, the finite quotient is fixed to  $\Gamma = \mathbb{Z}_6$ , so the realized gauge structure on the realized MAR-admissible branch is

$$\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}, \quad N_g = 3, \quad N_c = 3,$$

with the exact Standard Model hypercharge lattice on the realized matter package. Here the realized-branch counting chain is sequential: the asymptotic-freedom/CP-capability window and minimal admissibility first yield  $N_g = 3$ , and then Witten's SU(2) anomaly parity constraint together with the same realized-branch selector yield  $N_c = 3$ .

*Proof.* Items (i)–(iii) are collected from Theorem 4.3, Corollary 6.5, Theorem 6.19, and Corollary 6.20. The compact gauge reconstruction is Theorem 7.1, applied only after the transportable sector colimit and its faithful bosonic fiber functor are supplied by Assumption 3.7; the preceding MaxEnt/local-Gibbs/collar-mixing package is not used there to prove existence of that colimit or to decide whether it is trivial or nontrivial. The product gauge structure up to finite quotient is Theorem 7.9; and the final identification of the exact Standard Model quotient is Theorem 7.10 together with Corollaries 7.11, 7.12, and Proposition 7.13.  $\square$

Theorem 3.9 therefore packages only the recovered relativity-plus-Standard-Model core: D1–D5 together with D7–D9. The D6 capacity relation is a separate input-dependent corollary; the calibration sector and phenomenological continuations remain outside its scope.

## Usually Postulated Elsewhere, Recovered or Continued Here with Explicit Status

Structure	Common treatment	OPH treatment
Lorentz kinematics	background symmetry or starting axiom	conditional scaling-limit branch from geometric modular flow on screen caps
Einstein dynamics	fundamental field equation	conditional first-variation relation with tensor upgrade from generalized entropy, null modular data, and entanglement equilibrium
Gauge group	model input	conditional compact group reconstruction from edge sectors; exact SM quotient selected on the realized MAR-admissible branch
Hypercharge lattice	matter-assignment input	solved from anomaly cancellation and Yukawa invariance on the realized one-generation chiral matter plus one-Higgs package
Color and generation count	empirical input	fixed on the realized MAR-admissible branch by anomaly, CP capability, asymptotic freedom, and minimality
Flavor hierarchy unit	model-dependent small parameter	phenomenological ansatz $\varepsilon = 1/6$ motivated by a uniform $\mathbb{Z}_6$ center-label ensemble
Charged-lepton continuation	Koide-style phenomenological fit	exact centered readback plus an open centered-to-affine frontier above $\hat{C}_e^{\text{cand}}$ ; any $\delta = 2/9$ phase relation is continuation-only
Cosmological constant	local vacuum-energy puzzle	global screen-capacity term after the input-dependent capacity identification

## 4 Observer-Overlap Consistency as a Fixed-Point Problem

The discrete overlap problem is the mathematical skeleton of the framework.

**Definition 4.1** (Patch Net). *Let  $G = (V, E)$  be a finite connected graph. Each vertex  $i \in V$  carries a finite local state space  $S_i$ . For each edge  $e = \{i, j\}$  let  $I_e$  be an interface alphabet and let*

$$\pi_{i,e} : S_i \rightarrow I_e, \quad \pi_{j,e} : S_j \rightarrow I_e$$

*be interface projections. The global state space is*

$$\Sigma = \prod_{i \in V} S_i,$$

*and the consistency set is*

$$C = \{s \in \Sigma : \pi_{i,e}(s_i) = \pi_{j,e}(s_j) \text{ for all } e = \{i, j\} \in E\}.$$

This finiteness is a regulator-level assumption for the discrete patch-net normal-form theorem; it is not the continuum limit statement.

Define the inconsistency potential

$$\Phi(s) = \sum_{e=\{i,j\} \in E} w_e d_e(\pi_{i,e}(s_i), \pi_{j,e}(s_j)),$$

with  $w_e > 0$  and  $d_e(a, b) = 0$  iff  $a = b$ .

**Definition 4.2** (Recovery-derived local repair law). *A law  $\lambda$  is a family of local repair maps*

$$T_i^\lambda : \Sigma \rightarrow \Sigma$$

*changing only patch  $i$  or a bounded neighborhood of  $i$ . On the fixed-cutoff collar branch these are not free rewrite primitives: each accepted local update is read from exact Markov splice or a declared Petz/Fawzi–Renner recovery move on a collar chart around  $i$ , then lifted back to the finite patch presentation, and committed only when it strictly lowers the touched-overlap contribution to  $\Phi$  on the overlaps it can modify. Disjoint accepted repairs therefore commute directly by support locality, and when two accepted repairs touch a common overlap they are read inside one finite union collar whose physical glued state is parenthesization-independent on the quotient and whose local decoders are restriction-compatible on nested collars.*

**Theorem 4.3** (Asynchronous Fixed-Point Law). *Assume normal forms of the repair relation are exactly the globally consistent states  $C$ . Then every  $s \in \Sigma$  has a unique normal form*

$$\text{nf}_\lambda(s) \in C,$$

*and every maximal asynchronous repair sequence terminates at that same state.*

*Proof.* Each accepted local update changes only its touched-overlap set and is committed only when the sum of the corresponding  $\Phi$ -terms strictly decreases. Therefore every accepted repair strictly lowers the global inconsistency potential  $\Phi$ . Because  $\Sigma$  is finite, no infinite repair sequence exists, so the repair relation is terminating. If two accepted repairs are disjoint, support locality makes them commute directly. If they overlap, the same definition places them inside one parenthesization-independent union-collar gluing state on the physical quotient, so the two one-step branches have a common descendant. Thus the repair relation is locally confluent. Newman’s lemma [1] now gives confluence. The resulting normal form is unique. By assumption it is a globally consistent state.  $\square$

**Corollary 4.4** (Schedule Independence). *If observables are evaluated on  $\text{nf}_\lambda(s)$ , the resulting physical law is independent of update order.*

**Theorem 4.5** (Cycle Obstruction / Holonomy Criterion). *Let  $A$  be an abelian group. For each oriented edge  $e : u \rightarrow v$  assign  $b_e \in A$  and consider the affine overlap equations*

$$x_v - x_u = b_e.$$

*A global solution exists if and only if the signed sum of  $b_e$  vanishes on every cycle.*

*Proof.* Summing the edge equations around a cycle gives necessity. For sufficiency, fix a root and define each  $x_v$  by summing labels along a path from the root to  $v$ . Vanishing cycle sums make this independent of the chosen path.  $\square$

**Definition 4.6** (Physical repair law and representative lift). *Suppose a local gauge group  $\Gamma = \prod_i \Gamma_i$  acts on  $\Sigma$  while leaving all interface data invariant. Write*

$$q : \Sigma \rightarrow \Sigma/\Gamma, \quad q(s) = [s].$$

*A physical repair law is the family of local quotient maps induced by the recovery-derived collar updates just described:*

$$\bar{T}_i^\lambda : \Sigma/\Gamma \rightarrow \Sigma/\Gamma$$

*on the overlap-invariant quotient. A representative repair family is any family*

$$T_i^\lambda : \Sigma \rightarrow \Sigma$$

*with*

$$q \circ T_i^\lambda = \bar{T}_i^\lambda \circ q.$$

**Proposition 4.7** (Representative lifts descend to the quotient). *For any representative repair family of Definition 4.6,*

$$q(T_i^\lambda(\gamma \cdot s)) = q(T_i^\lambda(s)) \quad \forall i, \forall \gamma \in \Gamma, \forall s \in \Sigma.$$

*In particular the repair relation descends to  $\Sigma/\Gamma$  without any extra gauge-covariance axiom.*

*Proof.* Because  $q(\gamma \cdot s) = q(s)$ ,

$$q(T_i^\lambda(\gamma \cdot s)) = \bar{T}_i^\lambda(q(\gamma \cdot s)) = \bar{T}_i^\lambda(q(s)) = q(T_i^\lambda(s)).$$

So gauge-equivalent inputs induce the same repaired physical state.  $\square$

**Proposition 4.8** (Gauge Quotient). *Under Definition 4.6, Proposition 4.7, and Theorem 4.3, the normal-form map descends to the quotient:*

$$\bar{\text{nf}}_\lambda : \Sigma/\Gamma \rightarrow \Sigma/\Gamma, \quad [s] \mapsto [\text{nf}_\lambda(s)].$$

*Hence gauge-invariant observables are unique on the quotient, and the fixed-cutoff physical observable algebra has representative-independent terminal expectations on that carrier.*

*Proof.* By Proposition 4.7, every repair step has quotient image determined only by the current orbit. Iterating, the quotient image of any repair sequence depends only on the initial orbit. Since every maximal repair sequence from  $s$  terminates at the unique normal form  $\text{nf}_\lambda(s)$  by Theorem 4.3, the terminal orbit  $[\text{nf}_\lambda(s)]$  depends only on  $[s]$ . This makes  $\bar{\text{nf}}_\lambda$  well-defined.  $\square$

On the fixed-cutoff quantum lift, the same quotient-local carrier does more than fix the terminal orbit: for every declared fixed-cutoff physical algebra, the terminal expectation functional is the same on any two representative lifts whose regional collar data lie in one quotient-local glued state, even when the microscopic representatives differ by gauge or sector relabelings on that carrier.

This removes the extra gauge-covariance axiom and keeps repair on its actual quotient-local carrier: recovery dynamics on fixed-cutoff collars. On the declared fixed-cutoff branch, the remaining theorem-local burden is repair completeness, together with the Petz support/CPTP clause where that branch is used. The touched-overlap local-fit contract supplies Lyapunov  $\Phi$ -descent on accepted moves, and the support-local commutation and union-collar compatibility package belongs to the declared repair law itself. Stability of that package under refinement or branch change is a separate question.

**Definition 4.9** (Ancilla stabilization). *Fix a finite-cutoff OPH realization*

$$\mathfrak{U} = (\{\mathcal{H}_P, \mathcal{A}(P), \omega_P\}_{P \in \mathcal{P}}, \Gamma, T^\lambda).$$

*Choose finite-dimensional ancillary factors  $K_P$  with product state  $\eta = \bigotimes_{P \in \mathcal{P}} \eta_P$ . The associated ancilla stabilization is the realization*

$$\mathfrak{U}^\eta = (\{\mathcal{H}_P^\eta, \mathcal{A}^\eta(P), \omega_P^\eta\}_{P \in \mathcal{P}}, \Gamma, (T^\lambda)^\eta),$$

*with*

$$\mathcal{H}_P^\eta := \mathcal{H}_P \otimes K_P, \quad \mathcal{A}^\eta(P) := \mathcal{A}(P) \otimes \mathbf{1}_{K_P}, \quad \omega_P^\eta := \omega_P \otimes \eta_P,$$

*and lifted repair dynamics acting trivially on the ancillas.*

**Proposition 4.10** (Ancilla-stable UV underdetermination). *Let  $\mathfrak{U}^\eta$  be an ancilla stabilization of a finite-cutoff OPH realization  $\mathfrak{U}$ . Then:*

1. *observable expectations on the physical subalgebras are unchanged:*

$$\omega_P^\eta(a \otimes \mathbf{1}_{K_P}) = \omega_P(a) \quad \forall a \in \mathcal{A}(P);$$

2. *the interacting local MaxEnt branch on the physical subalgebra is unchanged, since*

$$\omega_{\ell_{UV}^\eta}(\lambda) = \omega_{\ell_{UV}}(\lambda) \otimes \eta;$$

3. *for every collar split  $A : B : D$ ,*

$$I(A : D \mid B)_{\rho \otimes \eta} = I(A : D \mid B)_\rho,$$

*so the Fawzi–Renner remainder  $r_{\text{FR}}(\varepsilon)$  and the collar Markov modulus  $\delta_{A:B:D}^{\text{M}}(\varepsilon)$  are unchanged;*

4. *if the ancillas are inert under repair, then*

$$\text{nf}_\lambda^\eta(s \otimes k) = \text{nf}_\lambda(s) \otimes k,$$

*so the quotient normal form on physical observables is identical;*

5. *if the ancillas carry only trivial neutral sectors, the MAR-selected realized sector package is unchanged.*

Hence  $\mathfrak{U}$  and  $\mathfrak{U}^\eta$  are OPH-indistinguishable although they are different microscopic regulator realizations.

*Proof.* Item 1 is immediate from the definition of  $\mathcal{A}^\eta(P)$  and  $\omega_P^\eta$ . Item 2 is the product-state form of the same fixed-cutoff MaxEnt branch. Item 3 follows from additivity of entropy for product ancillas, which cancels in the conditional mutual-information combination and therefore leaves both  $r_{\text{FR}}$  and  $\delta^{\text{M}}$  unchanged. Item 4 holds because the lifted repair maps do not act on the ancillary factors. Item 5 is immediate when the ancillas are neutral and carry no extra realized sector data.  $\square$

**Definition 4.11** (OPH-stable UV equivalence). *Two finite-cutoff realizations  $\mathfrak{U}$  and  $\mathfrak{U}'$  are OPH-stably equivalent, written*

$$\mathfrak{U} \sim_{\text{OPH}} \mathfrak{U}',$$

*if after finite ancilla stabilizations they are related by local gauge-covariant \*-isomorphisms intertwining the observable patch net, overlap maps, local states, and repair dynamics on the physical subalgebras.*

**Corollary 4.12** (Unique physical UV branch only modulo OPH-stable equivalence). *Under the OPH axiom language, the UV invariant determined by the theory is the class  $[\mathfrak{U}]_{\text{OPH}}$ , not a unique microscopic regulator presentation. Literal microscopic UV uniqueness is therefore not an OPH invariant.*

*Proof.* Propositions 4.7 and 4.8 fix the schedule-independent physical branch on the quotient, while Proposition 4.10 shows that inert ancillary refinements leave every OPH observable invariant. So the physical branch is fixed only modulo  $\sim_{\text{OPH}}$ , not at the level of one microscopic representative.  $\square$

These three results encode much of the eventual physical interpretation. Objectivity becomes confluence, gauge symmetry becomes quotient invariance induced by the physical overlap algebra, and stable defects are represented by nontrivial overlap holonomy classes. The physically relevant uniqueness statement is therefore quotient uniqueness together with ancilla-stable equivalence: OPH fixes a unique gauge-invariant physical branch modulo boundary redundancy, implementation hiding, and inert ancillary stabilization, not a unique microscopic regulator presentation.

## 5 Collars, Edge Centers, and Generalized Entropy

The fixed-point picture explains why overlap consistency produces gauge quotients: once repair is read on the physical overlap algebra, descent to the quotient is automatic. The gravity and consensus arguments later use a more specific collar fact: after edge-center completion, interior observables become insensitive to compatible exterior substitutions, and modular additivity becomes exact in the Markov normal form. The point of this section is therefore twofold. First, at fixed regulator scale, we derive the collar block decomposition from overlap consistency itself on the ordinary or central-defect branch and then state its genuinely noncentral higher-gauge replacement. Second, we state precisely when the approximate recoverability premise of Axiom 3.4 is allowed to converge to the exact HJPW normal form used by the later spatial and null collar theorems.

The sharp claim boundary is as follows. Small conditional mutual information always gives a constructive recovered comparison state with  $O(\varepsilon^{1/2})$  observable error. It does *not* by itself give a universal one-shot trace-norm bound to an exact Markov state. The exact Markov normal form is recovered either when  $I(A : D \mid B) = 0$  holds literally, or for a controlled family on one fixed finite-dimensional collar model, or after pullback to such a model, where  $\varepsilon \rightarrow 0$ . On that fixed collar one gets a collar-dependent modulus  $\delta_{A:B:D}^{\text{M}}(\varepsilon) \rightarrow 0$  measuring distance to the exact Markov set, and this is the error that must be carried into later exact splice or modular-additivity identities.

**Proposition 5.1** (Derived regulator gluing datum). *Fix a regulator-scale finite patch cover of a cap neighborhood and let  $R$  be any finite union of regulator cells. Assume only the finite patch-net presentation implicit in Definition 4.1 together with overlap consistency on common interfaces. Then:*

1. *each regulator cell  $i$  with finite local state space  $S_i$  can be Hilbertized as  $\tilde{\mathcal{H}}_i \cong \mathbb{C}^{|S_i|}$ , so the extended algebra before overlap quotienting is the finite type-I algebra*

$$\tilde{\mathcal{A}}(R) = \mathcal{B}(\tilde{\mathcal{H}}_R), \quad \tilde{\mathcal{H}}_R := \bigotimes_{i \subset R} \tilde{\mathcal{H}}_i;$$

2. *for each connected boundary component  $\Sigma \subset \partial R$ , after choosing a reference cut presentation  $\tilde{\mathcal{H}}_\Sigma$ , every overlap-consistent change of local chart along  $\Sigma$  is implemented by a unitary on  $\tilde{\mathcal{H}}_\Sigma$ , and the compact closure of the subgroup generated by all such recharting unitaries is a compact boundary redundancy group*

$$K_\Sigma \subset U(\tilde{\mathcal{H}}_\Sigma);$$

*before the choice of reference chart, the same data form a compact unitary groupoid of overlap-preserving transitions;*

3. *if triple-overlap defects are central, the projective composition law of item 2 lifts to an direct action of a compact central extension  $\widehat{K}_\Sigma$ ; a genuinely noncentral defect is the only obstruction to reducing the transition system to an ordinary compact group action;*
4. *in the invariant-state realization of the quotient, the chart-independent endomorphisms of the lifted presentation form the boundary-invariant algebra*

$$\mathcal{A}_{\text{inv}}(R) = \tilde{\mathcal{A}}(R)^{\widehat{K}_{\partial R}}, \quad \widehat{K}_{\partial R} := \prod_{\Sigma \subset \partial R} \widehat{K}_\Sigma,$$

*with the convention  $\widehat{K}_\Sigma = K_\Sigma$  whenever no central extension is needed; the physical state space is the corresponding invariant subspace, and on collars the sector-preserving algebra induced on that subspace is the block-diagonal algebra used in Theorem 5.2.*

*Items 1 and the lifted fixed-point presentation of item 4 are the fixed-cutoff realized data used later; they are not extra axioms beyond the finite regulator presentation plus overlap redundancy.*

*Proof sketch.* Item 1 is the Hilbertization of finite sets. For item 2, each overlap-consistent recharting acts on a finite-dimensional matrix algebra, and every \*-automorphism of such an algebra is inner. After transporting all local overlap presentations to a chosen reference chart, each recharting is therefore implemented by a unitary on the cut Hilbert space. The subgroup generated by those unitaries has compact closure inside a finite-dimensional unitary group, which yields  $K_\Sigma$ ; without fixing the reference chart one has the corresponding compact unitary groupoid. Item 3 is the usual projective-versus-central-extension lift for a central 2-cocycle. Item 4 records the lifted fixed-point presentation: chart-independent endomorphisms of the unreduced boundary data are exactly the fixed points of the derived boundary action, while the later collar theorem uses the invariant-state realization and the sector-preserving algebra induced on the matched-cut subspace. If the defect is genuinely noncentral, the quotient is still well defined, but the correct bookkeeping object is the crossed-module or higher-gauge data rather than an ordinary compact group.  $\square$

**Theorem 5.2** (Derived edge-center collar decomposition and exact Markov normal form). *Let  $A$ - $B$ - $D$  be a collar tripartition realized at fixed regulator scale on the ordinary or central-defect branch of Proposition 5.1. Write  $B = B_L \cup B_R$  with common interface  $\Sigma = \partial C$ , and let  $\widehat{K}_\Sigma$  be the derived compact boundary action attached to that cut, with  $\widehat{K}_\Sigma = K_\Sigma$  on the ordinary branch. In the invariant-state realization,*

$$\mathcal{H}_B = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\widehat{K}_\Sigma}.$$

Decompose the left boundary data as

$$\tilde{\mathcal{H}}_{B_L} \cong \bigoplus_{\alpha} W_{\alpha} \otimes \mathcal{H}_{b_L^{\alpha}},$$

with  $W_{\alpha}$  irreducible  $\widehat{K}_\Sigma$ -modules. Because the right half-collar carries the inverse overlap transport across the same cut, it decomposes contragrediently:

$$\tilde{\mathcal{H}}_{B_R} \cong \bigoplus_{\beta} W_{\beta}^* \otimes \mathcal{H}_{b_R^{\beta}}.$$

Then

$$\mathcal{H}_B \cong \bigoplus_{\alpha} \mathcal{H}_{b_L^{\alpha}} \otimes \mathcal{H}_{b_R^{\alpha}},$$

and the sector-preserving collar algebra

$$\mathcal{A}_{\text{EC}}(B) := \bigoplus_{\alpha} \mathcal{B}(\mathcal{H}_{b_L^{\alpha}}) \otimes \mathcal{B}(\mathcal{H}_{b_R^{\alpha}})$$

has center

$$Z(\mathcal{A}_{\text{EC}}(B)) = \bigoplus_{\alpha} \mathbb{C} \mathbf{1}_{\alpha}.$$

If, in addition, the reference state on  $A \cup B \cup D$  is exact Markov,

$$I(A : D|B)_{\rho} = 0,$$

then the state decomposes as

$$\rho_{ABD} = \bigoplus_{\alpha} p_{\alpha} \rho_{Ab_L^{\alpha}}^{(\alpha)} \otimes \rho_{b_R^{\alpha}D}^{(\alpha)}.$$

The same formula holds in any limit state that is exact Markov on this fixed collar model. Thus the collar-center block decomposition is forced by overlap consistency plus the derived regulator package on the ordinary or central-defect branch; exact Markov factorization is an additional state condition, not a consequence of the block decomposition alone.

*Proof.* By Proposition 5.1, one may realize the physical collar states as the  $\widehat{K}_\Sigma$ -invariant subspace of  $\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R}$ . Complete reducibility of finite-dimensional unitary representations gives the displayed decomposition of  $\tilde{\mathcal{H}}_{B_L}$ , and the right half-collar carries the inverse transport law across the same cut, hence the contragredient decomposition of  $\tilde{\mathcal{H}}_{B_R}$ . Therefore

$$\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R} \cong \bigoplus_{\alpha, \beta} (W_{\alpha} \otimes W_{\beta}^*) \otimes (\mathcal{H}_{b_L^{\alpha}} \otimes \mathcal{H}_{b_R^{\beta}}).$$

Taking  $\widehat{K}_\Sigma$ -invariants gives

$$\mathcal{H}_B \cong \bigoplus_{\alpha, \beta} (W_{\alpha} \otimes W_{\beta}^*)^{\widehat{K}_\Sigma} \otimes (\mathcal{H}_{b_L^{\alpha}} \otimes \mathcal{H}_{b_R^{\beta}}).$$

By Schur's lemma,

$$(W_\alpha \otimes W_\beta^*)^{\widehat{K}_\Sigma} \cong \begin{cases} \mathbb{C}, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

which yields the displayed block decomposition. The sector-preserving collar algebra therefore has the displayed block-diagonal form, so its center is generated by the block projectors. Within each block, observables from  $A \cup B$  act only on the left factor and observables from  $B \cup D$  act only on the right factor. If the state is exact Markov, the standard HJPW structure theorem gives the blockwise factorized normal form [15]. The same formula is used in the idealized recoverability limit when exact identities are taken literally.  $\square$

**Theorem 5.3** (Derived higher-gauge cut datum). *Fix a connected cut  $\Sigma$  and a finite regulator chart  $\{P_i\}_{i \in I_\Sigma}$  meeting along  $\Sigma$ , with finite nerve  $N_\Sigma$ . On the genuinely noncentral branch, the weak gluing data are implemented by a compact crossed module*

$$\mathbb{K}_\Sigma = (H_\Sigma \xrightarrow{\partial_\Sigma} G_\Sigma, \triangleright)$$

together with maps

$$g_{ij} : P_{ij} \rightarrow G_\Sigma, \quad h_{ijk} : P_{ijk} \rightarrow H_\Sigma,$$

obeying

$$g_{ij}g_{jk} = \partial_\Sigma(h_{ijk})g_{ik}, \quad h_{jkl}h_{ijl} = (g_{ij} \triangleright h_{ikl})h_{ijk}.$$

The corresponding higher-gauge change system on the cut is the compact semidirect product

$$\mathcal{T}_\Sigma := C^1(N_\Sigma, H_\Sigma) \rtimes C^0(N_\Sigma, G_\Sigma),$$

with multiplication

$$(\eta, u) \cdot (\eta', u') = (\eta(u \triangleright \eta'), uu'),$$

and standard coboundary action

$$\begin{aligned} g_{ij} &\mapsto u_i \partial_\Sigma(\eta_{ij}) g_{ij} u_j^{-1}, \\ h_{ijk} &\mapsto (u_i \triangleright h_{ijk}) \eta_{ij} (g_{ij} \triangleright \eta_{jk}) \eta_{ik}^{-1}. \end{aligned}$$

The fixed-cutoff physical sector on the genuinely noncentral branch is therefore the orbit class

$$q_\Sigma = [(g, h)] \in \check{H}^2(N_\Sigma, H_\Sigma \rightarrow G_\Sigma),$$

and no external continuous input  $P$  or  $N_{\text{scr}}$  enters this theorem package.

*Proof sketch.* On pair overlaps, every regulator-scale recharting acts on a finite-dimensional matrix algebra and is therefore inner, so it is implemented by a unitary on the cut Hilbert space. On triple overlaps, genuinely noncentral failure of strict composition is recorded by unitary associators rather than scalars. These 1- and 2-morphisms form a compact unitary 2-group of rechartings. Skeletal strictification of a compact 2-group is equivalent to crossed-module data, which yields the displayed compact  $\mathbb{K}_\Sigma$ . Because the nerve is finite,  $C^1(N_\Sigma, H_\Sigma)$  and  $C^0(N_\Sigma, G_\Sigma)$  are finite products of compact groups, hence  $\mathcal{T}_\Sigma$  is compact. The displayed formulas are the standard crossed-module coboundary action, and the orbit class is exactly the nonabelian Čech 2-class of the defect data; see [27] for an explicit higher-lattice-gauge realization of the same crossed-module structures.  $\square$

**Theorem 5.4** (Higher-gauge edge-center completion). *Let  $B = B_L \cup B_R$  be a fixed-cutoff collar around a connected cut  $\Sigma$  on the genuinely noncentral branch of Theorem 5.3. Then, as unitary  $\mathcal{T}_\Sigma$ -modules,*

$$\tilde{\mathcal{H}}_{B_L} \cong \bigoplus_{\lambda \in \Lambda_\Sigma} W_\lambda \otimes \mathcal{H}_{b_L^\lambda}, \quad \tilde{\mathcal{H}}_{B_R} \cong \bigoplus_{\mu \in \Lambda_\Sigma} W_\mu^* \otimes \mathcal{H}_{b_R^\mu},$$

for a finite set  $\Lambda_\Sigma$  of irreducible unitary representations of  $\mathcal{T}_\Sigma$  that actually occur. Hence the physical higher-gauge collar space is

$$\mathcal{H}_B^{2g} := (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\mathcal{T}_\Sigma} \cong \bigoplus_{\lambda \in \Lambda_\Sigma} \mathcal{H}_{b_L^\lambda} \otimes \mathcal{H}_{b_R^\lambda},$$

and the collar algebra and its center are

$$\mathcal{A}_{2g}(B) \cong \bigoplus_{\lambda \in \Lambda_\Sigma} \mathcal{B}(\mathcal{H}_{b_L^\lambda}) \otimes \mathcal{B}(\mathcal{H}_{b_R^\lambda}), \quad Z(\mathcal{A}_{2g}(B)) = \bigoplus_{\lambda \in \Lambda_\Sigma} \mathbb{C} \mathbf{1}_\lambda.$$

*Proof sketch.* Because  $\mathcal{T}_\Sigma$  is compact and the half-collar spaces are finite-dimensional, both  $\tilde{\mathcal{H}}_{B_L}$  and  $\tilde{\mathcal{H}}_{B_R}$  split orthogonally into irreducible unitary  $\mathcal{T}_\Sigma$ -modules with finite multiplicity. The right side sees inverse transport across the same cut, hence the contragredient module. Therefore

$$\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R} \cong \bigoplus_{\lambda, \mu} (W_\lambda \otimes W_\mu^*) \otimes (\mathcal{H}_{b_L^\lambda} \otimes \mathcal{H}_{b_R^\mu}).$$

Taking  $\mathcal{T}_\Sigma$ -invariants leaves

$$(W_\lambda \otimes W_\mu^*)^{\mathcal{T}_\Sigma} \cong \text{Hom}_{\mathcal{T}_\Sigma}(W_\mu, W_\lambda) \cong \begin{cases} \mathbb{C}, & \lambda = \mu, \\ 0, & \lambda \neq \mu, \end{cases}$$

by Schur's lemma. This yields the matched block decomposition and the center formula.  $\square$

**Theorem 5.5** (Higher-gauge Markov collar and carried errors). *Let  $\rho_{ABD}$  be a faithful state on a collar whose middle region  $B$  carries the higher-gauge block decomposition of Theorem 5.4. If*

$$I(A : D \mid B)_\rho = 0,$$

then

$$\rho_{ABD} = \bigoplus_{\lambda \in \Lambda_\Sigma} p_\lambda \rho_{Ab_L^\lambda} \otimes \rho_{b_R^\lambda D}.$$

If instead  $I(A : D \mid B)_\rho \leq \varepsilon$  on one fixed faithful collar model, then the same carried Fawzi–Renner and fixed-collar Markov errors as in the ordinary branch continue to hold:

$$r_{\text{FR}}(\varepsilon) = 2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}, \quad 4\lambda_*^{-1} \delta_{A:B:D}^{\text{M}}(\varepsilon).$$

*Proof sketch.* Theorem 5.4 shows that the higher-gauge collar algebra is still a finite direct sum of type-I tensor blocks. HJPW depends only on that finite-dimensional algebraic structure, not on whether the block label arose from an ordinary compact group, a central extension, or a crossed-module gauge system. Likewise, the Fawzi–Renner remainder and the fixed-collar Markov modulus are finite-dimensional entropy and recovery statements and are blind to the origin of the block label.  $\square$

**Theorem 5.6** (Higher-gauge defect transportability). *With the genuinely noncentral cut data of Theorem 5.3, define*

$$q_\Sigma := [(g, h)] \in \check{H}^2(N_\Sigma, H_\Sigma \rightarrow G_\Sigma).$$

Then:

1.  $q_\Sigma$  is invariant under all local rechartings and higher-gauge changes  $(\eta, u) \in \mathcal{T}_\Sigma$ .
2. Two genuinely noncentral representatives describe the same fixed-cutoff physical cut sector if and only if they determine the same class  $q_\Sigma$ .
3. The defect is removable if and only if  $q_\Sigma = 0$ .

Thus the genuinely noncentral branch carries the same transportability logic as the ordinary or central branch, but in crossed-module Čech degree 2 rather than ordinary group cohomology.

*Proof sketch.* Items 1–3 are the crossed-module version of ordinary Čech transportability: coboundaries change representatives while preserving classes, the trivial class strictifies the cocycle, and a nontrivial class cannot be gauged away. In any explicit crossed-module lattice realization, the same sectors are equivalently the connected components of the 2-flat configuration groupoid, or homotopy classes of maps into the classifying space  $B\mathbb{K}_\Sigma$ .  $\square$

**Proposition 5.7** (Higher-gauge interacting compatibility). *In a fixed-cutoff crossed-module lattice realization of Theorem 5.3, one may add local collar terms*

$$K_{\text{collar}}^{2g} = \sum_{v \in \Sigma_0} \beta_v(1 - A_v) + \sum_{e \in \Sigma_1} \beta_e(1 - B_e) + \sum_{c \in \mathcal{C}_\Sigma} \beta_c(1 - C_c),$$

where  $A_v$ ,  $B_e$ , and  $C_c$  are the local gauge and fake-flat projectors of the chosen higher-gauge realization. These terms are bounded-support and commute in the standard higher-gauge projector realization [27], so the full fixed-cutoff generator remains quasi-local and inside the Axiom 3.3 class.

*Proof sketch.* The additional terms are local projector constraints supported on finitely many collar cells, so they preserve bounded support and quasi-locality. In the standard higher-gauge projector realization they commute, so the same finite-constraint MaxEnt/Lieb–Robinson bookkeeping used on the ordinary branch continues to apply.  $\square$

**Remark 5.8** (Status of the regulator package and the precise Markov boundary). *Proposition 5.1, Theorem 5.2, and Theorems 5.3–5.6 together with Proposition 5.7 show that edge-center completion is a theorem of overlap consistency plus the finite regulator presentation on the ordinary, central-defect, and genuinely noncentral higher-gauge branches. No preferred quantum-link realization is assumed on the ordinary or central branch, and the genuinely noncentral branch is handled by the compact crossed-module replacement rather than by one more ordinary-group lemma. The fixed-cutoff topological UV package is closed on all three branches. The continuum modular/geometric lift on the geometric subnet under **T2** and, whenever refinement-limit compact gauge reconstruction is invoked, the bosonic ordinary/central gauge premises of Assumption 3.7, form the current boundary. The distinct issue addressed next is state control: exact splice and modular-additivity identities require either literal exact Markovity or a controlled family that converges to the exact Markov set on one fixed finite-dimensional collar.*

**Definition 5.9** (Exact Markov set and collar-distance modulus). *Fix the finite-dimensional collar Hilbert space of Theorem 5.2. Let*

$$\mathfrak{M}_{A:B:D} := \{\sigma_{ABD} : I(A : D | B)_\sigma = 0\}$$

*be the set of exact Markov states on that fixed collar. Equivalently, by HJPW and Theorem 5.2,  $\sigma \in \mathfrak{M}_{A:B:D}$  iff*

$$\sigma_{ABD} = \bigoplus_{\alpha} q_{\alpha} \sigma_{Ab_L^{\alpha}}^{(\alpha)} \otimes \sigma_{b_R^{\alpha}D}^{(\alpha)}$$

*for the fixed edge-center decomposition of  $B$ .*

*For a general state  $\rho_{ABD}$ , define its distance to the exact Markov set by*

$$d_M(\rho) := \inf_{\sigma \in \mathfrak{M}_{A:B:D}} \|\rho - \sigma\|_1,$$

*and the fixed-collar exact-Markov modulus by*

$$\delta_{A:B:D}^M(\varepsilon) := \sup \{d_M(\rho) : I(A : D | B)_\rho \leq \varepsilon\}.$$

**Proposition 5.10** (Controlled exact Markov-collar limit). *On a fixed finite-dimensional collar, the exact Markov set  $\mathfrak{M}_{A:B:D}$  is compact and*

$$\delta_{A:B:D}^M(\varepsilon) \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

*Consequently, if  $\rho_{ABD}^{(n)}$  is any sequence of states on that same collar with*

$$I(A : D | B)_{\rho^{(n)}} \leq \varepsilon_n, \quad \varepsilon_n \rightarrow 0,$$

*then there exist exact Markov states  $\sigma^{(n)} \in \mathfrak{M}_{A:B:D}$  such that*

$$\|\rho^{(n)} - \sigma^{(n)}\|_1 \leq \delta_{A:B:D}^M(\varepsilon_n) \longrightarrow 0.$$

*Thus approximate recoverability converges to the exact HJPW normal form used later.*

*Proof.* The full state space on a fixed finite-dimensional Hilbert space is compact in trace norm. Conditional mutual information is continuous there because von Neumann entropy is continuous in finite dimension. Hence  $\mathfrak{M}_{A:B:D}$  is closed, so it is compact as a closed subset of a compact space.

Suppose  $\delta_{A:B:D}^M(\varepsilon) \not\rightarrow 0$ . Then there exist  $\eta > 0$ , a sequence  $\varepsilon_n \downarrow 0$ , and states  $\rho^{(n)}$  with  $I(A : D | B)_{\rho^{(n)}} \leq \varepsilon_n$  but  $d_M(\rho^{(n)}) \geq \eta$  for all  $n$ . By compactness, pass to a trace-norm convergent subsequence  $\rho^{(n_k)} \rightarrow \rho^*$ . Continuity of conditional mutual information gives

$$I(A : D | B)_{\rho^*} = \lim_{k \rightarrow \infty} I(A : D | B)_{\rho^{(n_k)}} = 0,$$

so  $\rho^* \in \mathfrak{M}_{A:B:D}$ . But then

$$d_M(\rho^{(n_k)}) \leq \|\rho^{(n_k)} - \rho^*\|_1 \longrightarrow 0,$$

contradicting  $d_M(\rho^{(n_k)}) \geq \eta$ . Thus  $\delta_{A:B:D}^M(\varepsilon) \rightarrow 0$ . The final statement follows by choosing  $\sigma^{(n)} \in \mathfrak{M}_{A:B:D}$  within  $\delta_{A:B:D}^M(\varepsilon_n) + 1/n$  of  $\rho^{(n)}$ .  $\square$

**Proposition 5.11** (Approximate collar recovery). *Let  $A$ - $B$ - $D$  be a collar tripartition satisfying*

$$I(A : D|B)_\rho \leq \varepsilon,$$

*and let  $\mathcal{R}_{B \rightarrow BD}$  be a recovery map such that*

$$F(\rho_{ABD}, (\text{id}_A \otimes \mathcal{R}_{B \rightarrow BD})(\rho_{AB})) \geq e^{-\varepsilon/2}.$$

*Define the recovered comparison state*

$$\rho_{ABD}^{\text{rec}} := (\text{id}_A \otimes \mathcal{R}_{B \rightarrow BD})(\rho_{AB}), \quad r_{\text{FR}}(\varepsilon) := 2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}.$$

*Then*

$$\|\rho_{ABD} - \rho_{ABD}^{\text{rec}}\|_1 \leq r_{\text{FR}}(\varepsilon).$$

*Consequently, for every CPTP map  $\Lambda_{BD \rightarrow B'D'}$  and every bounded observable  $X$  supported on  $A \cup B'$ ,*

$$|\text{Tr}[X(\text{id}_A \otimes \Lambda)(\rho_{ABD} - \rho_{ABD}^{\text{rec}})]| \leq \|X\|_\infty \|(\text{id}_A \otimes \Lambda)(\rho_{ABD} - \rho_{ABD}^{\text{rec}})\|_1 \leq \|X\|_\infty r_{\text{FR}}(\varepsilon).$$

*Proof.* The fidelity lower bound and the Fuchs–van de Graaf inequality give

$$\|\rho_{ABD} - \rho_{ABD}^{\text{rec}}\|_1 \leq 2\sqrt{1 - F(\rho_{ABD}, \rho_{ABD}^{\text{rec}})} \leq 2\sqrt{1 - e^{-\varepsilon}} = r_{\text{FR}}(\varepsilon).$$

The estimate  $r_{\text{FR}}(\varepsilon) \leq 2\sqrt{\varepsilon}$  follows from  $1 - e^{-x} \leq x$ . Contractivity of trace norm under CPTP maps gives the transported bound, and the observable estimate is the duality between trace norm and operator norm [14]. The bound is constructive and dimension-free. What it controls is closeness to a recovered comparison state; no claim is made that  $\rho_{ABD}^{\text{rec}}$  is itself exact Markov.  $\square$

**Proposition 5.12** (Exact splice theorem and controlled collar approximation). *Under Theorem 5.2, let  $\sigma_{ABD} \in \mathfrak{M}_{A:B:D}$  have exact Markov form*

$$\sigma_{ABD} = \bigoplus_{\alpha} p_{\alpha} \sigma_{Ab_L^{\alpha}}^{(\alpha)} \otimes \sigma_{b_R^{\alpha}D}^{(\alpha)}.$$

*Let  $\tau_{b_R^{\alpha}D'}^{(\alpha)}$  be any normalized family of states compatible with the same right-boundary sectors, and define*

$$\sigma'_{ABD'} = \bigoplus_{\alpha} p_{\alpha} \sigma_{Ab_L^{\alpha}}^{(\alpha)} \otimes \tau_{b_R^{\alpha}D'}^{(\alpha)}.$$

*Define the common left algebra*

$$\mathcal{A}_{Ab_L} := \bigoplus_{\alpha} \mathcal{B}(\mathcal{H}_{Ab_L^{\alpha}}) \otimes \mathbf{1}_{b_R^{\alpha}},$$

*canonically represented on both direct-sum Hilbert spaces. Then:*

1. *for every  $X \in \mathcal{A}_{Ab_L}$ ,*

$$\text{Tr}(X\sigma'_{ABD'}) = \text{Tr}(X\sigma_{ABD});$$

2. *if  $\rho_{ABD}$  is any state on the same fixed collar with  $I(A : D | B)_\rho \leq \varepsilon$ , and if  $\sigma_{\varepsilon} \in \mathfrak{M}_{A:B:D}$  is chosen so that*

$$\|\rho_{ABD} - \sigma_{\varepsilon}\|_1 \leq \delta_{A:B:D}^{\text{M}}(\varepsilon),$$

*then the corresponding exact splice  $\sigma'_{\varepsilon}$  satisfies*

$$|\text{Tr}(X\rho_{ABD}) - \text{Tr}(X\sigma'_{\varepsilon})| \leq \|X\|_\infty \delta_{A:B:D}^{\text{M}}(\varepsilon)$$

*for all  $X \in \mathcal{A}_{Ab_L}$ .*

Hence the exact splice identity used later is justified either at exact Markovity or along any controlled collar family for which  $\delta_{A:B:D}^M(\varepsilon_\delta) \rightarrow 0$ .

*Proof.* For item 1, blockwise factorization gives

$$\mathrm{Tr}(X\sigma'_{ABD'}) = \sum_{\alpha} p_{\alpha} \mathrm{Tr}\left(X_{\alpha}\sigma_{Ab_L^{\alpha}}^{(\alpha)}\right) \mathrm{Tr}\left(\tau_{b_R^{\alpha}D'}^{(\alpha)}\right).$$

Here  $X_{\alpha}$  denotes the  $\alpha$ -block of  $X$  in  $\mathcal{A}_{Ab_L}$ . Each  $\tau^{(\alpha)}$  is normalized, so the second factor is 1. The same computation for  $\sigma_{ABD}$  gives the same value because the original right factor is likewise normalized.

For item 2, apply item 1 to  $\sigma_{\varepsilon}$ :

$$\mathrm{Tr}(X\sigma'_{\varepsilon}) = \mathrm{Tr}(X\sigma_{\varepsilon}).$$

Therefore

$$|\mathrm{Tr}(X\rho_{ABD}) - \mathrm{Tr}(X\sigma'_{\varepsilon})| = |\mathrm{Tr}[X(\rho_{ABD} - \sigma_{\varepsilon})]| \leq \|X\|_{\infty} \|\rho_{ABD} - \sigma_{\varepsilon}\|_1 \leq \|X\|_{\infty} \delta_{A:B:D}^M(\varepsilon).$$

□

**Proposition 5.13** (Transport of modular-additivity errors from the exact Markov reference). *Let  $K_X(\omega) := -\log \omega_X$  for a faithful reduced state on region  $X$ , and define the modular defect*

$$\Delta K(\omega) := K_{ABD}(\omega) - K_{AB}(\omega) - K_{BD}(\omega) + K_B(\omega).$$

*Fix a collar and let  $\rho$  be a faithful state with  $I(A : D | B)_{\rho} \leq \varepsilon$ . Choose  $\sigma_{\varepsilon} \in \mathfrak{M}_{A:B:D}$  with*

$$\|\rho - \sigma_{\varepsilon}\|_1 \leq \delta_{A:B:D}^M(\varepsilon).$$

*Assume that the four marginals  $\rho_X$  and  $(\sigma_{\varepsilon})_X$  for  $X \in \{ABD, AB, BD, B\}$  all satisfy*

$$\rho_X \geq \lambda_* \mathbf{1}, \quad (\sigma_{\varepsilon})_X \geq \lambda_* \mathbf{1}$$

*on their supports for some  $\lambda_* > 0$ . Then*

$$\|\Delta K(\rho) - \Delta K(\sigma_{\varepsilon})\|_{\infty} \leq 4\lambda_*^{-1} \delta_{A:B:D}^M(\varepsilon).$$

*Since  $\Delta K(\sigma_{\varepsilon})$  is central or blockwise constant, every matrix element of the modular defect of  $\rho$  is within  $4\lambda_*^{-1} \delta_{A:B:D}^M(\varepsilon)$  of the exact Markov value.*

*Proof.* By monotonicity of trace distance under partial trace,

$$\|\rho_X - (\sigma_{\varepsilon})_X\|_1 \leq \|\rho - \sigma_{\varepsilon}\|_1 \leq \delta_{A:B:D}^M(\varepsilon)$$

for each  $X \in \{ABD, AB, BD, B\}$ . On the interval  $[\lambda_*, 1]$ , the function  $\log x$  has derivative bounded by  $\lambda_*^{-1}$ . By the standard integral representation of the operator logarithm, this implies the operator-norm Lipschitz bound

$$\|K_X(\rho) - K_X(\sigma_{\varepsilon})\|_{\infty} = \|\log(\sigma_{\varepsilon})_X - \log \rho_X\|_{\infty} \leq \lambda_*^{-1} \|\rho_X - (\sigma_{\varepsilon})_X\|_{\infty} \leq \lambda_*^{-1} \delta_{A:B:D}^M(\varepsilon).$$

Summing the four region contributions gives

$$\|\Delta K(\rho) - \Delta K(\sigma_{\varepsilon})\|_{\infty} \leq 4\lambda_*^{-1} \delta_{A:B:D}^M(\varepsilon).$$

For  $\sigma_{\varepsilon} \in \mathfrak{M}_{A:B:D}$ , the exact HJPW block factorization implies that  $\Delta K(\sigma_{\varepsilon})$  is central or blockwise constant, so the same bound controls all matrix elements relative to the exact Markov modular-additivity value. □

**Remark 5.14** (What is and is not carried forward). *There are therefore two distinct quantitative controls, and this paper keeps them separate:*

1. *the constructive one-shot recoverability bound  $r_{\text{FR}}(\varepsilon) = O(\varepsilon^{1/2})$ , which compares the physical state to a recovered comparison state and is enough for bounded-observable statements at one regulator stage;*
2. *the fixed-collar exact-Markov modulus  $\delta_{A:B:D}^{\text{M}}(\varepsilon) \rightarrow 0$ , which compares the physical state to the exact Markov set and is the correct quantity for justifying exact splice or modular-additivity identities in a controlled limit, with an additional factor  $4\lambda_*^{-1}$  when logarithms are involved.*

*Later spatial and null collar theorems therefore use exact identities only in two regimes: literal exact Markovity, or a controlled refinement or scaling family in which the relevant  $\delta^{\text{M}}(\varepsilon_\delta)$  tends to zero after transport to one fixed finite-dimensional collar model. The manuscript does not claim a universal dimension-free one-shot trace-norm estimate from small conditional mutual information directly to an exact Markov state.*

**Remark 5.15** (Floor transfer to the exact-Markov reference). *On one fixed finite-dimensional collar model, Proposition 5.13 does not require a second independent faithfulness input on the exact-Markov comparison family. Suppose the transported physical marginals satisfy*

$$\rho_X \geq \bar{\lambda}_{m,\delta} \mathbf{1}$$

*eventually for each  $X \in \{ABD, AB, BD, B\}$ , and choose exact-Markov replacements  $\sigma_{n,m,\delta}$  with*

$$\|\rho_X - (\sigma_{n,m,\delta})_X\|_1 \leq \delta_{m,\delta}^{\text{M}}(\varepsilon_{n,m,\delta}) \longrightarrow 0.$$

*Then  $\|\rho_X - (\sigma_{n,m,\delta})_X\|_\infty \leq \|\rho_X - (\sigma_{n,m,\delta})_X\|_1$ , so for all sufficiently large  $n$ ,*

$$(\sigma_{n,m,\delta})_X \geq \frac{1}{2} \bar{\lambda}_{m,\delta} \mathbf{1}.$$

*Thus the only nonlatent faithfulness-side input below the modular-defect term is the eventual lower spectral bound on the transported marginals themselves; the exact-Markov reference inherits a common floor once the exact-Markov modulus tends to zero.*

**Proposition 5.16** (Generalized-entropy split). *If the reduced cap state takes the block form*

$$\rho_C = \bigoplus_{\alpha} p_{\alpha} \left( \rho_{\text{bulk},C}^{(\alpha)} \otimes \frac{\mathbf{1}_{\text{edge}}^{(\alpha)}}{d_{\alpha}} \right),$$

*then*

$$S(\rho_C) = S_{\text{bulk}}(C) + \text{Tr}(\rho_C L_C),$$

*where*

$$S_{\text{bulk}}(C) = H(p_{\alpha}) + \sum_{\alpha} p_{\alpha} S(\rho_{\text{bulk},C}^{(\alpha)}), \quad L_C = \sum_{\alpha} (\log d_{\alpha}) P_{\alpha}.$$

*Proof.* The entropy of a direct sum is

$$S\left(\bigoplus_{\alpha} p_{\alpha} \sigma_{\alpha}\right) = H(p_{\alpha}) + \sum_{\alpha} p_{\alpha} S(\sigma_{\alpha}).$$

Apply this identity with

$$\sigma_\alpha = \rho_{\text{bulk},C}^{(\alpha)} \otimes \frac{\mathbf{1}_{\text{edge}}^{(\alpha)}}{d_\alpha}.$$

Since

$$S(\sigma_\alpha) = S(\rho_{\text{bulk},C}^{(\alpha)}) + \log d_\alpha,$$

the first statement follows.  $\square$

**Definition 5.17** (Effective Newton constant in the refinement dictionary). *In the refinement-scaling regime, if*

$$\text{Tr}(\rho_C L_C) \approx N_\Sigma \bar{\ell}(t), \quad A(\partial C) \approx N_\Sigma a_{\text{cell}},$$

then matching  $\text{Tr}(\rho_C L_C)$  to the area term  $A(\partial C)/(4G)$  defines the effective Newton constant

$$G := \frac{a_{\text{cell}}}{4\bar{\ell}(t)}.$$

*This is a continuum dictionary identification, not part of the exact algebraic proposition.*

Proposition 5.12 is the algebraic version of interior invariance under compatible exterior substitutions. Proposition 5.13 is the precise bridge from small collar conditional mutual information to the exact modular identities later used by the spatial and null branches: at finite stage one carries a remainder, and exact identities appear only when that remainder is zero or tends to zero in the controlled collar limit. Proposition 5.16 isolates the exact entropy split, while Definition 5.17 supplies the continuum identification that turns the edge center into a gravitational coupling.

## 6 Modular Geometry, Relativity, and Einstein Dynamics

### 6.1 Geometric modular flow

Before stating the continuum modular theorem, fix the algebraic target. At fixed cutoff, the full operational patch algebra contains geometric overlap data, declared record summaries, and compare/write/verify pointer layers. The BW question should be asked only on the overlap-generated geometric subnet, not on the entire operational algebra.

**Definition 6.1** (Operational total algebra, geometric subnet, and auxiliary/record sector). *At fixed cutoff  $n$  and patch  $P$ , let  $\mathcal{A}_n^{\text{tot}}(P)$  denote the operational patch algebra consisting of the physical patch algebra together with the declared overlap-readable summaries used by the compare/write/verify surface. Let  $B_{IJ} \subset P$  range over overlap collars incident on  $P$ , let  $\Pi_\alpha^{(IJ)}$  denote the overlap sector projectors, and let  $Q_a^{(IJ)}$  denote the boundary observables carrying the geometric cut data. Define the geometric subnet by*

$$\mathcal{A}_n^{\text{geo}}(P) := W^* \left\langle \bigcup_{B_{IJ} \subset P} \iota_{IJ \rightarrow P}(\{\Pi_\alpha^{(IJ)}\}_\alpha \cup \{Q_a^{(IJ)}\}_a) \right\rangle_{\text{repair, isotony}} / \mathcal{N}_n(P),$$

where  $\mathcal{N}_n(P)$  is the maximal repair-invariant overlap-trivial kernel, namely the maximal part whose restriction to every overlap algebra inside  $P$  is scalar. The complementary operational sector generated by record summaries, pointer registers, and other interface-inert observables is denoted  $\mathcal{A}_n^{\text{aux/rec}}(P)$ .

**Theorem 6.2** (Geometric subnet and operational primeness). *For each fixed-cutoff patch  $P$ , the operational algebra admits a canonical quotient*

$$\pi_{n,P}^{\text{geo}} : \mathcal{A}_n^{\text{tot}}(P) \twoheadrightarrow \mathcal{A}_n^{\text{geo}}(P)$$

with the following properties:

- (i)  $\mathcal{A}_n^{\text{geo}}(P)$  is generated by the overlap sector projectors and geometric boundary observables, and is closed under repair and isotony;
- (ii) the record-summary block, pointer algebra, and interface-inert auxiliary observables lie in the complementary operational sector  $\mathcal{A}_n^{\text{aux/rec}}(P)$  and do not enlarge  $\mathcal{A}_n^{\text{geo}}(P)$ ;
- (iii) if  $N \subset \mathcal{A}_n^{\text{geo}}(P)$  is a repair-invariant subalgebra whose restriction to every overlap algebra inside  $P$  is scalar, then  $N = \mathbb{C}\mathbf{1}$ .

*Proof.* Items (i) and (ii) are the content of Definition 6.1. For item (iii), if such an  $N$  were nontrivial, then its pullback under  $\pi_{n,P}^{\text{geo}}$  would define a nontrivial repair-invariant overlap-trivial component of  $\mathcal{A}_n^{\text{tot}}(P)$ , contradicting the maximality of  $\mathcal{N}_n(P)$ . Thus no nontrivial overlap-trivial factor survives inside  $\mathcal{A}_n^{\text{geo}}(P)$ .  $\square$

The collar analysis proves a fixed-cutoff statement on the finite type-I regulator net: the reduced cap state has a literal density matrix, its modular Hamiltonian exists, and its nonadditive part is confined to a shrinking collar up to carried errors. The Lorentz claim is therefore not a literal statement about the finite regulator matrices. Under the controlled scaling clause **T2**, the target object is a realized scaling-limit geometric cap pair

$$(\mathcal{A}_\infty^{\text{geo}}(C), \omega_\infty^{\text{geo},C})$$

for each round cap  $C \subset S^2$ . Axiom 3.3 controls the state-side realized branch across refinement through one common finite-dimensional MaxEnt family, and Theorem 6.2 removes the overlap-trivial auxiliary/record layers from the BW target. What still remains theorem-local on the present UV surface is not a separate global branch selector but two explicit objects on that geometric subnet: a realized geometric cap-pair extraction from transported marginals, and ordered cut-pair rigidity on the extracted prime cap pair. The theorem below makes that claim boundary explicit, fixes the  $2\pi$  normalization once those two objects are supplied, and leaves the remaining UV burden exposed rather than hidden.

Fix a cap  $C \subset S^2$  and a shrinking collar family  $(A_\delta, B_\delta, D_\delta)$  around  $\partial C$ , with  $\delta \downarrow 0$  and  $A_\delta \cup B_\delta \cup D_\delta$  covering a neighborhood of the cut. Write

$$\varepsilon_\delta := I(A_\delta : D_\delta \mid B_\delta)_\omega, \quad r_{\text{FR}}(\varepsilon_\delta) := 2\sqrt{1 - e^{-\varepsilon_\delta}} \leq 2\sqrt{\varepsilon_\delta},$$

and, on each fixed faithful collar model,

$$\eta_\delta^{\text{M}} := 4\lambda_*^{-1} \delta_{A_\delta : B_\delta : D_\delta}^{\text{M}}(\varepsilon_\delta),$$

where  $\lambda_* > 0$  is the lower spectral bound used to compare modular Hamiltonians. Every exact collar identity below is therefore to be read in one of two ways:

1. literal exactness when the reference collar state is exact Markov; or

2. a controlled collar family for which

$$\delta_{A_\delta:B_\delta:D_\delta}^M(\varepsilon_\delta) \rightarrow 0 \quad (\delta \downarrow 0),$$

with the finite-stage errors  $r_{\text{FR}}(\varepsilon_\delta)$  and  $\eta_\delta^M$  carried explicitly.

For each cap  $C$ , let  $\lambda_C(s) \subset \text{Conf}^+(S^2)$  denote the standard cap-preserving conformal one-parameter subgroup, normalized so that the null blow-up near a smooth cut acts by  $v \mapsto e^{-s}v$ , and let  $\alpha_{\lambda_C(s)}$  denote the induced automorphism of the scaling-limit cap net.

**Theorem 6.3** (Geometric modular flow on caps on the extracted prime geometric subnet). *Assume Axioms 3.1–3.4, Theorem 6.2, the derived fixed-cutoff regulator/collar package established above, and the controlled scaling-limit scope clause T2 of Assumption 3.6. Assume in addition that for each round cap  $C \subset S^2$ :*

(a) *the transported cap marginals on  $\mathcal{A}_n^{\text{geo}}(C_n)$  admit a realized scaling-limit geometric cap pair*

$$(\mathcal{A}_\infty^{\text{geo}}(C), \omega_\infty^{\text{geo},C})$$

*through the carried-collar local weak-\* / GNS extraction package; and*

(b) *ordered cut-pair rigidity holds on that extracted prime geometric pair, so the cap-preserving conformal subgroup singled out by the cut data is unique up to normalization.*

*Then:*

(i) *At each finite regulator stage, the cap algebra is type I, the reduced cap state has a density matrix  $\rho_C^{(\delta)}$ , and its modular Hamiltonian*

$$K_C^{(\delta)} := -\log \rho_C^{(\delta)}$$

*has nonadditive part confined to the shrinking collar up to the carried errors measured by  $r_{\text{FR}}(\varepsilon_\delta)$  and  $\eta_\delta^M$ . In particular, on each fixed collar model the physical collar state differs from its constructive recovered comparison state by  $r_{\text{FR}}(\varepsilon_\delta)$  in trace norm, and its modular defect differs from the exact-Markov reference defect by at most  $\eta_\delta^M$ .*

(ii) *For every realized scaling-limit geometric cap pair  $(\mathcal{A}_\infty^{\text{geo}}(C), \omega_\infty^{\text{geo},C})$  furnished by hypothesis (a), the modular automorphism group is geometric:*

$$\sigma_t^{\omega_\infty^{\text{geo},C}} = \alpha_{\lambda_C(2\pi t)}.$$

*Equivalently, the modular parameter  $t$  and the geometric cap-dilation parameter  $s$  are related by*

$$s = 2\pi t.$$

(iii) *No separate cap-isotropy/SO(2)-equivariance selector or Euclidean-regularity premise is used in this theorem. Once ordered cut-pair rigidity has fixed the geometric cap subgroup on the extracted prime geometric pair and  $\lambda_C(s)$  is normalized by the null blow-up  $v \mapsto e^{-s}v$ , the modular KMS normalization fixes the coefficient  $2\pi$ .*

(iv) *If the scaling-limit cap algebra happens to remain type I, item (ii) may be written as the operator identity*

$$K_C = 2\pi B_C.$$

*In the generic continuum case, where the scaling-limit cap algebra has left the regulator class and is expected in QFT examples to be non-type-I, the correct theorem statement is item (ii) itself, and the geometric modular action is generally outer rather than inner.*

*Consequently, for every controlled collar family with*

$$\delta_{A_\delta: B_\delta: D_\delta}^M(\varepsilon_\delta) \rightarrow 0$$

*and surviving faithfulness bound, replacing the finite-stage modular action by the geometric cap-dilation action incurs only the carried errors  $r_{\text{FR}}(\varepsilon_\delta)$  and  $\eta_\delta^M$ , and these vanish in the refinement limit.*

*Proof.* Step 1: fixed-cutoff collar control. On the finite type-I regulator net, Propositions 5.12, 5.13, and 5.16 localize the modular defect to the shrinking collar and quantify the discrepancy between the physical collar state, its constructive recovered comparison state, and its exact-Markov reference by the carried errors  $r_{\text{FR}}(\varepsilon_\delta)$  and  $\eta_\delta^M$ .

Step 2: scaling-limit claim tier. Theorem 6.3 is a scaling-limit statement by **T2**, so the theorem concerns only scaling-limit cap pairs furnished by the controlled refinement/scaling package, not the finite regulator matrices themselves. Nothing in this step assumes that type-I structure survives the limit.

Step 3: geometric identification on the prime subnet. By hypothesis (a) and Theorem 6.2, the realized limit pair is extracted on the overlap-generated prime geometric subnet, with overlap-trivial auxiliary/record factors removed. By hypothesis (b), the cap-preserving conformal action singled out by the ordered cut data is unique up to normalization. Therefore there is a constant  $\kappa_C > 0$  such that

$$\sigma_t^{\omega_\infty^{\text{geo}, C}} = \alpha_{\lambda_C(\kappa_C t)}.$$

Step 4: normalization. Because ordered cut-pair rigidity has identified the standard cap-preserving conformal subgroup on the extracted geometric pair, no separate cap-isotropy or Euclidean-regularity selector enters. The modular group is KMS at inverse temperature 1 by definition. Since  $\lambda_C(s)$  is normalized by unit null dilation under the blow-up  $v \mapsto e^{-s}v$ , the Bisognano–Wichmann normalization fixes

$$\kappa_C = 2\pi,$$

hence

$$\sigma_t^{\omega_\infty^{\text{geo}, C}} = \alpha_{\lambda_C(2\pi t)}.$$

Step 5: operator versus automorphism form. If the limit cap algebra remains type I, one can represent the modular automorphism group by a modular Hamiltonian  $K_C = -\log \rho_C$  and recover  $K_C = 2\pi B_C$ . If the limit algebra leaves the regulator class, the modular automorphism group remains well defined while the inner operator representative need not exist inside  $\mathcal{A}_\infty^{\text{geo}}(C)$ . In that case the automorphism identity is the full theorem statement. The final sentence is exactly the carried-error conclusion from Step 1 transported along the controlled scaling family of **T2**.  $\square$

**Remark 6.4** (BW-side UV scaffold after the geometric-subnet split). *Theorem 6.2 removes the old ambiguity about the target algebra: the continuum Lorentz claim is asked only on the extracted prime*

geometric subnet, not on the full operational algebra. The UV-side boundary is therefore a two-step chain on  $\mathcal{A}_n^{\text{geo}}$ , and neither step is an independent cap-isotropy selector or a Euclidean-regularity premise.

First, one needs a realized transported geometric cap-local system for caps and their conformal images. The relevant package is the cap-local test family on  $\mathcal{A}_n^{\text{geo}}(C_n)$ , the projectively compatible transported marginal family, and the asymptotic transport-equivalence certificate. On each fixed local collar model  $(m, \delta)$ , let  $\varepsilon_{n,m,\delta}$  be the transported collar conditional mutual information. If the transported physical geometric marginals satisfy an eventual common floor

$$\rho_X^{(n,m,\delta)} \geq \bar{\lambda}_{m,\delta} \mathbf{1} \quad (X \in \{ABD, AB, BD, B\})$$

and

$$\delta_{A_{n,m,\delta}:B_{n,m,\delta}:D_{n,m,\delta}}^{\text{M}}(\varepsilon_{n,m,\delta}) \rightarrow 0,$$

then the exact-Markov comparison marginals inherit the same floor for all sufficiently large  $n$ . The carried-collar schedule

$$\eta_{n,m,\delta}^{\text{cap}} := r_{\text{FR}}(\varepsilon_{n,m,\delta}) + 4\bar{\lambda}_{m,\delta}^{-1} \delta_{A_{n,m,\delta}:B_{n,m,\delta}:D_{n,m,\delta}}^{\text{M}}(\varepsilon_{n,m,\delta})$$

is therefore derived rather than primitive. The only nonlatent lower input on the faithfulness side is the eventual fixed-local-collar modular-transport common floor on the transported physical marginals themselves; no second comparison-state faithfulness clause is missing.

Second, once local weak-\* extraction and GNS gluing have emitted the realized scaling-limit geometric cap pair, one needs ordered cut-pair rigidity on that realized pair to collapse the residual cap-preserving conformal freedom to the standard hyperbolic subgroup. Only after that second object is supplied does the geometric modular statement become fully internal on  $\mathcal{A}_{\infty}^{\text{geo}}(C)$ . The present paper proves neither object and therefore does not claim that MaxEnt alone completes the BW lift.

**Corollary 6.5** (Lorentz Kinematics). *Under the hypotheses of Theorem 6.3,*

$$\text{Conf}^+(S^2) \cong \text{PSL}(2, \mathbb{C}) \cong \text{SO}^+(3, 1).$$

The cap modular flows are therefore the standard one-parameter Lorentz boost/dilation subgroups in the celestial-sphere realization, and the induced local kinematic group is the connected Lorentz group.

*Proof.* Orientation-preserving conformal maps of  $S^2$  are exactly the Möbius transformations, so

$$\text{Conf}^+(S^2) \cong \text{PSL}(2, \mathbb{C}) \cong \text{SO}^+(3, 1).$$

By Theorem 6.3, on the extracted prime geometric subnet the realized scaling-limit cap modular automorphism groups are the standard cap-preserving conformal dilations with the  $2\pi$  normalization fixed internally. Hence the local kinematics induced by modular flow is the connected Lorentz group.  $\square$

## 6.2 The null modular bridge

The fixed-cutoff strip analysis begins from the transferred cut-center data and the inherited-strip Markov structure. At fixed regulator scale one obtains exact or controlled four-term strip additivity together with endpoint-Lipschitz control of the renormalized half-line family; on the same OPH geometric scaling branch used in Theorem 6.3, the null half-line blow-up net then inherits

geometric dilation and therefore half-sided modular inclusion. Standard Borchers–Wiesbrock theory [6] consequently applies to the resulting OPH half-sided modular pair, yielding positivity and unitary implementation of null translations. What remains downstream is narrower: the separate interval-preserving projective branch is still needed to transport the half-line result to bounded null intervals, and the later tensor upgrade still carries the standard null-invisible metric ambiguity. The half-line generator itself is identified below with the effective local null-stress charge.

The present subsection therefore establishes the following chain:

1. null-strip center transfer and the inherited left/right split hypothesis;
2. exact strip additivity on the exact inherited Markov model, and a carried defect operator on controlled exact-Markov replacements on one fixed strip model;
3. endpoint-Lipschitz matrix elements for the renormalized half-line family and the resulting weak tail generator;
4. derived half-sided modular inclusion on the null half-line blow-up net from OPH scaling-limit geometry, and the resulting Borchers positive translation generator.

The only explicit downstream boundary retained below is the bounded-interval transport/projective branch together with the later tensor upgrade; the half-line generator/charge identification is proved inside the bridge itself. Lemma 6.12 states the positive null-translation generator explicitly: the Borchers unitary group is constructed on its Stone domain, positivity is theorem-level rather than implicit, and the half-line modular Hamiltonians are tied to it by explicit affine covariance and quadratic-form identities. The density-upgrade template of Theorem 6.10 is recorded separately and does not carry the operator-identification burden.

**Proposition 6.6** (Null-strip center transfer and inherited split). *Fix a null generator  $\Omega$  and a regulated tripartition*

$$I_- = (v_1, v_2), \quad J = (v_2, v_3), \quad I_+ = (v_3, v_4),$$

with cuts

$$\Gamma_- := \{v = v_2\}, \quad \Gamma_+ := \{v = v_3\}.$$

Assume the derived fixed-cutoff regulator/collar package established above, and assume that the two null cuts inherit the ordinary or central-defect boundary-redundancy data of Proposition 5.1 in the following precise sense:

- (i) each cut  $\Gamma_\pm$  carries a compact derived boundary action  $\widehat{K}_\pm$  on a reference cut Hilbert space, with  $\widehat{K}_\pm = K_\pm$  on the ordinary branch;
- (ii) in a compatible type-I regulator presentation, the adjacent regions carry inverse transport across each cut, so for irreducible  $\widehat{K}_\pm$ -modules  $W_{\alpha_\pm}$  one has decompositions

$$\begin{aligned} \tilde{\mathcal{H}}_{I_-} &\cong \bigoplus_{\alpha_-} W_{\alpha_-} \otimes \mathcal{H}_{i_-^{\alpha_-}}, \\ \tilde{\mathcal{H}}_J &\cong \bigoplus_{\alpha_-, \alpha_+} W_{\alpha_-}^* \otimes \mathcal{H}_{j^{\alpha_-, \alpha_+}} \otimes W_{\alpha_+}, \\ \tilde{\mathcal{H}}_{I_+} &\cong \bigoplus_{\alpha_+} W_{\alpha_+}^* \otimes \mathcal{H}_{i_+^{\alpha_+}}; \end{aligned}$$

(iii) the gauge-invariant strip algebra is the commutant of the transported boundary action on  $\tilde{\mathcal{H}}_J$ , so that the pair  $(\alpha_-, \alpha_+)$  is a central cut label;

(iv) for each  $(\alpha_-, \alpha_+)$ , the multiplicity space  $\mathcal{H}_{j^{\alpha_-, \alpha_+}}$  admits a chosen factorization

$$\mathcal{H}_{j^{\alpha_-, \alpha_+}} \cong \mathcal{H}_{j_L^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{j_R^{\alpha_-, \alpha_+}}$$

such that, blockwise,  $\mathcal{A}(I_- \cup J)$  acts only on  $\mathcal{H}_{i_-^{\alpha_-}} \otimes \mathcal{H}_{j_L^{\alpha_-, \alpha_+}}$  and  $\mathcal{A}(J \cup I_+)$  acts only on  $\mathcal{H}_{j_R^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{i_+^{\alpha_+}}$ .

Then the strip algebra has the inherited central decomposition

$$\mathcal{A}(J) \cong \bigoplus_{\alpha_-, \alpha_+} \mathcal{B}(\mathcal{H}_{j^{\alpha_-, \alpha_+}}), \quad Z(\mathcal{A}(J)) = \bigoplus_{\alpha_-, \alpha_+} \mathbb{C} P_{\alpha_-, \alpha_+},$$

and, under item (iv),

$$\mathcal{A}(J) \cong \bigoplus_{\alpha_-, \alpha_+} \mathcal{B}(\mathcal{H}_{j_L^{\alpha_-, \alpha_+}}) \otimes \mathcal{B}(\mathcal{H}_{j_R^{\alpha_-, \alpha_+}}).$$

Moreover the glued tripartition carries the block decomposition

$$\mathcal{H}_{I_- \cup J \cup I_+} \cong \bigoplus_{\alpha_-, \alpha_+} \mathcal{H}_{i_-^{\alpha_-}} \otimes \mathcal{H}_{j_L^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{j_R^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{i_+^{\alpha_+}}.$$

Thus the two null cuts transfer the same center data as the spatial collar branch, while the left/right split of the strip multiplicity spaces is exactly the extra decomposition-inheritance hypothesis and is not forced by center transfer alone.

*Proof.* By complete reducibility of the finite-dimensional unitary actions  $\widehat{K}_\pm$ , the displayed decompositions of  $\tilde{\mathcal{H}}_{I_-}$ ,  $\tilde{\mathcal{H}}_J$ , and  $\tilde{\mathcal{H}}_{I_+}$  exist. Under item (iii), the strip algebra is the commutant of the transported  $\widehat{K}_- \times \widehat{K}_+$  action on

$$\tilde{\mathcal{H}}_J = \bigoplus_{\alpha_-, \alpha_+} W_{\alpha_-}^* \otimes \mathcal{H}_{j^{\alpha_-, \alpha_+}} \otimes W_{\alpha_+}.$$

Write an operator  $X \in \mathcal{B}(\tilde{\mathcal{H}}_J)$  in matrix form relative to that direct sum. The block

$$X_{(\alpha_-, \alpha_+), (\beta_-, \beta_+)}$$

intertwines

$$W_{\alpha_-}^* \otimes W_{\alpha_+} \longrightarrow W_{\beta_-}^* \otimes W_{\beta_+}$$

for the product action of  $\widehat{K}_- \times \widehat{K}_+$ . By Schur's lemma, such an intertwiner is zero unless  $(\alpha_-, \alpha_+) = (\beta_-, \beta_+)$ , and when the sector pair agrees it is scalar on the representation factors. Therefore the commutant is exactly

$$\mathcal{A}(J) \cong \bigoplus_{\alpha_-, \alpha_+} \mathcal{B}(\mathcal{H}_{j^{\alpha_-, \alpha_+}}),$$

and the center is generated by the corresponding block projectors  $P_{\alpha_-, \alpha_+}$ .

For the glued tripartition, tensor the three regulator Hilbert spaces and decompose:

$$\tilde{\mathcal{H}}_{I_-} \otimes \tilde{\mathcal{H}}_J \otimes \tilde{\mathcal{H}}_{I_+} \cong \bigoplus_{\alpha_-, \beta_-, \alpha_+, \beta_+} (W_{\alpha_-} \otimes W_{\beta_-}^*) \otimes \mathcal{H}_{i_-^{\alpha_-}} \otimes \mathcal{H}_{j^{\beta_-, \beta_+}} \otimes (W_{\beta_+} \otimes W_{\alpha_+}^*) \otimes \mathcal{H}_{i_+^{\alpha_+}}.$$

Taking  $\widehat{K}_- \times \widehat{K}_+$ -invariants and applying Schur's lemma to each cut gives

$$(W_{\alpha_-} \otimes W_{\beta_-}^*)^{\widehat{K}_-} \cong \begin{cases} \mathbb{C}, & \alpha_- = \beta_-, \\ 0, & \alpha_- \neq \beta_-, \end{cases} \quad (W_{\beta_+} \otimes W_{\alpha_+}^*)^{\widehat{K}_+} \cong \begin{cases} \mathbb{C}, & \beta_+ = \alpha_+, \\ 0, & \beta_+ \neq \alpha_+, \end{cases}$$

so only matching cut sectors survive and one obtains

$$\mathcal{H}_{I_- \cup J \cup I_+} \cong \bigoplus_{\alpha_-, \alpha_+} \mathcal{H}_{i_-^{\alpha_-}} \otimes \mathcal{H}_{j^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{i_+^{\alpha_+}}.$$

If item (iv) holds, substitute the chosen factorization

$$\mathcal{H}_{j^{\alpha_-, \alpha_+}} \cong \mathcal{H}_{j_L^{\alpha_-, \alpha_+}} \otimes \mathcal{H}_{j_R^{\alpha_-, \alpha_+}}$$

to obtain the displayed inherited split and the stated action of the left and right union algebras. Nothing in the Schur-lemma argument itself forces item (iv); it is exactly the extra structural condition required to reproduce the same blockwise tensor pattern as Theorem 5.2.  $\square$

**Corollary 6.7** (Exact or controlled four-term null modular relation on an inherited strip model). *Fix one finite-dimensional regulated strip model satisfying Proposition 6.6, and define*

$$\mathfrak{M}_{I_-:J:I_+} := \{\sigma : I(I_- : I_+ | J)_\sigma = 0\},$$

$$\delta_{I_-:J:I_+}^M(\varepsilon) := \sup \left\{ \inf_{\sigma \in \mathfrak{M}_{I_-:J:I_+}} \|\rho - \sigma\|_1 : I(I_- : I_+ | J)_\rho \leq \varepsilon \right\}.$$

For any state  $\eta$  on this strip model, write

$$\Delta K_J(\eta) := K_{I_- \cup J \cup I_+}(\eta) - K_{I_- \cup J}(\eta) - K_{J \cup I_+}(\eta) + K_J(\eta).$$

If the reference state  $\omega$  is exact Markov on this inherited decomposition, then

$$\Delta K_J(\omega) \in Z(\mathcal{A}(J)),$$

and in the canonical blockwise identification of Proposition 6.6 one may take

$$\Delta K_J(\omega) = 0.$$

Equivalently, without fixing that normalization there exists a central operator

$$K_{\partial,J}(\omega) \in Z(\mathcal{A}(J))$$

such that

$$K_{I_- \cup J \cup I_+}(\omega) = K_{I_- \cup J}(\omega) + K_{J \cup I_+}(\omega) - K_J(\omega) + K_{\partial,J}(\omega).$$

If instead  $I(I_- : I_+ | J)_\omega \leq \varepsilon$ , then on this fixed strip model one may choose an exact Markov state  $\tilde{\omega}_J \in \mathfrak{M}_{I_-:J:I_+}$  with

$$\|\omega - \tilde{\omega}_J\|_1 \leq \delta_{I_-:J:I_+}^M(\varepsilon).$$

Define the carried defect operator

$$\mathfrak{D}_J(\omega, \tilde{\omega}_J) := \Delta K_J(\omega) - \Delta K_J(\tilde{\omega}_J).$$

Then

$$K_{I_- \cup J \cup I_+}(\omega) = K_{I_- \cup J}(\omega) + K_{J \cup I_+}(\omega) - K_J(\omega) + K_{\partial, J}(\tilde{\omega}_J) + \mathfrak{D}_J(\omega, \tilde{\omega}_J),$$

where

$$K_{\partial, J}(\tilde{\omega}_J) := \Delta K_J(\tilde{\omega}_J) \in Z(\mathcal{A}(J)).$$

For every bounded observable  $O$  in  $\mathcal{A}(I_- \cup J)$  or  $\mathcal{A}(J \cup I_+)$ ,

$$|\omega(O) - \tilde{\omega}_J(O)| \leq \|O\|_\infty \delta_{I_-:J:I_+}^M(\varepsilon).$$

If the marginals entering  $\Delta K_J$  for  $\omega$  and  $\tilde{\omega}_J$  are uniformly faithful with lower spectral bound  $\lambda_* > 0$ , then

$$\|\mathfrak{D}_J(\omega, \tilde{\omega}_J)\|_\infty \leq 4\lambda_*^{-1} \delta_{I_-:J:I_+}^M(\varepsilon).$$

Independently, the Fawzi–Renner recovery map supplies a constructive comparison state with trace-norm error

$$r_{\text{FR}}(\varepsilon) = 2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}.$$

Hence the exact four-term strip relation is available only at exact Markovity, or along a controlled strip family on one fixed inherited strip model for which

$$\delta_{I_-:J:I_+}^M(\varepsilon_J) \rightarrow 0.$$

In the controlled case,  $\mathfrak{D}_J$  is carried explicitly at each finite stage.

*Proof.* Assume first that  $\omega$  is exact Markov on the inherited split of Proposition 6.6. Then the HJPW structure theorem gives, on each sector pair  $\alpha = (\alpha_-, \alpha_+)$ ,

$$\omega_{I_- J I_+}|_\alpha = p_\alpha \omega_{I_- j_L}^{(\alpha)} \otimes \omega_{j_R I_+}^{(\alpha)}.$$

Accordingly,

$$K_{I_- \cup J \cup I_+}(\omega)|_\alpha = (-\log p_\alpha) \mathbf{1} + K_{I_- j_L}^{(\alpha)} \otimes \mathbf{1}_{j_R I_+} + \mathbf{1}_{I_- j_L} \otimes K_{j_R I_+}^{(\alpha)}.$$

Tracing over  $I_+$ ,  $I_-$ , or both gives the marginal block formulas

$$K_{I_- \cup J}(\omega)|_\alpha = (-\log p_\alpha) \mathbf{1} + K_{I_- j_L}^{(\alpha)} \otimes \mathbf{1}_{j_R} + \mathbf{1}_{I_- j_L} \otimes K_{j_R}^{(\alpha)},$$

$$K_{J \cup I_+}(\omega)|_\alpha = (-\log p_\alpha) \mathbf{1} + K_{j_L}^{(\alpha)} \otimes \mathbf{1}_{j_R I_+} + \mathbf{1}_{j_L} \otimes K_{j_R I_+}^{(\alpha)},$$

$$K_J(\omega)|_\alpha = (-\log p_\alpha) \mathbf{1} + K_{j_L}^{(\alpha)} \otimes \mathbf{1}_{j_R} + \mathbf{1}_{j_L} \otimes K_{j_R}^{(\alpha)}.$$

Subtracting the last three expressions from the first gives zero on every block. Thus, in the canonical blockwise identification supplied by Proposition 6.6,

$$\Delta K_J(\omega) = 0.$$

If one chooses to keep the blockwise endpoint-label bookkeeping explicit rather than absorbing those constants into the canonical identification, the remainder is a direct sum of block constants and is therefore central. This yields the stated central form with  $K_{\partial, J}(\omega) \in Z(\mathcal{A}(J))$ .

For the controlled statement, apply Proposition 5.10 on this fixed finite-dimensional strip model, relabeling  $A, B, D$  there as  $I_-, J, I_+$ . This yields an exact Markov replacement  $\tilde{\omega}_J \in \mathfrak{M}_{I_-:J:I_+}$  with the displayed trace-norm bound. The identity

$$\Delta K_J(\omega) = \Delta K_J(\tilde{\omega}_J) + \mathfrak{D}_J(\omega, \tilde{\omega}_J)$$

is tautological, so the displayed four-term formula follows from the exact-Markov case applied to  $\tilde{\omega}_J$ . The observable estimate is immediate from

$$|\omega(O) - \tilde{\omega}_J(O)| \leq \|O\|_\infty \|\omega - \tilde{\omega}_J\|_1.$$

Finally, Proposition 5.13, again relabeled  $A, B, D \mapsto I_-, J, I_+$ , yields the operator-norm bound on  $\mathfrak{D}_J$  whenever the relevant marginals have the stated lower spectral bound. The Fawzi–Renner estimate is the same one-shot recoverability bound used elsewhere and is independent of the exact-Markov replacement modulus.  $\square$

**Definition 6.8** (Renormalized null modular functional). *For a null interval  $I$  on generator  $\Omega$ , let  $K_\partial(I, \Omega)$  denote the central endpoint-label term singled out by Corollary 6.7 on the inherited strip model, or by its controlled exact-Markov replacement when that model is used as reference. Define*

$$\widetilde{K}[I, \Omega] := K[I, \Omega] - K_\partial(I, \Omega).$$

For a null half-line  $H_a := (a, \infty)$ , abbreviate

$$\widetilde{K}_a(\Omega) := \widetilde{K}[H_a, \Omega].$$

**Proposition 6.9** (Endpoint-Lipschitz null modular families and weak tail generator). *Under the derived quasi-local propagation and bounded-interval endpoint control internal to Axiom 3.3, the renormalized interval family obeys the following matrix-element bounds on every fixed local-energy-bounded domain. For bounded intervals*

$$I = (a, b), \quad I' = (a', b'),$$

contained in one compact endpoint window,

$$|\langle \psi, (\widetilde{K}[I', \Omega] - \widetilde{K}[I, \Omega])\phi \rangle| \leq C_{\psi, \phi, \Omega} (|a' - a| + |b' - b|).$$

In particular the matrix elements of  $\widetilde{K}[I, \Omega]$  are jointly Lipschitz in the two endpoints and therefore have finite variation there.

For null half-lines  $H_a = (a, \infty)$ , one likewise has

$$|\langle \psi, (\widetilde{K}_{a'}(\Omega) - \widetilde{K}_a(\Omega))\phi \rangle| \leq C_{\psi, \phi, \Omega} |a' - a|$$

for  $a, a'$  in every compact window. Hence, for fixed  $\psi, \phi$ ,

$$f_{\psi, \phi}(a) := \langle \psi, \widetilde{K}_a(\Omega)\phi \rangle$$

is locally Lipschitz and therefore absolutely continuous. Its distributional derivative defines a locally  $L^\infty$  sesquilinear-form density

$$\langle \psi, q(a, \Omega)\phi \rangle := -\frac{1}{2\pi} \partial_a f_{\psi, \phi}(a),$$

and for  $a < b$ ,

$$\langle \psi, (\widetilde{K}_b(\Omega) - \widetilde{K}_a(\Omega))\phi \rangle = -2\pi \int_a^b \langle \psi, q(v, \Omega)\phi \rangle dv.$$

The object  $q(a, \Omega)$  is the weak tail generator. At this stage it is not yet asserted to be represented by a local operator-valued density; its identification with the positive self-adjoint Borchers generator occurs only after the derived half-sided modular inclusion of Corollary 6.11 and Lemma 6.12.

*Proof.* The bounded-interval endpoint-control statement derived from Axiom 3.3 applies precisely to the renormalized family obtained after removal of the central endpoint term. Therefore, for intervals contained in one fixed compact endpoint window,

$$|\langle \psi, (\widetilde{K}[I', \Omega] - \widetilde{K}[I, \Omega])\phi \rangle| \leq C_{\psi, \phi, \Omega} |I' \Delta I|,$$

where  $I' \Delta I$  is the symmetric difference of the two intervals. If

$$I = (a, b), \quad I' = (a', b'),$$

then

$$|I' \Delta I| \leq |a' - a| + |b' - b|,$$

which gives the displayed joint endpoint-Lipschitz bound.

For half-lines  $H_a = (a, \infty)$  and  $H_{a'} = (a', \infty)$ , the symmetric difference is the bounded interval between the two endpoints, so

$$|H_{a'} \Delta H_a| = |a' - a|.$$

Because the central endpoint-label term has been removed in the definition of  $\widetilde{K}_a$ , the same endpoint-control estimate yields

$$|\langle \psi, (\widetilde{K}_{a'}(\Omega) - \widetilde{K}_a(\Omega))\phi \rangle| \leq C_{\psi, \phi, \Omega} |a' - a|.$$

Hence, on every compact  $a$ -interval, the function

$$f_{\psi, \phi}(a) := \langle \psi, \widetilde{K}_a(\Omega)\phi \rangle$$

is Lipschitz. A Lipschitz function on a compact interval is absolutely continuous, so it has an almost-everywhere derivative in  $L^\infty$  and obeys the fundamental theorem of calculus:

$$f_{\psi, \phi}(b) - f_{\psi, \phi}(a) = \int_a^b \partial_v f_{\psi, \phi}(v) dv.$$

Define

$$\langle \psi, q(v, \Omega)\phi \rangle := -\frac{1}{2\pi} \partial_v f_{\psi, \phi}(v)$$

distributionally. Substituting this definition into the previous identity gives

$$\langle \psi, (\widetilde{K}_b(\Omega) - \widetilde{K}_a(\Omega))\phi \rangle = -2\pi \int_a^b \langle \psi, q(v, \Omega)\phi \rangle dv.$$

This is exactly the claimed weak-tail formula.  $\square$

**Theorem 6.10** (Downstream density-upgrade template). *Fix a null generator  $\Omega$  and the renormalized half-line family  $\widetilde{K}_a(\Omega)$  of Proposition 6.9. For fixed vectors  $\psi, \phi$  in a common dense domain, let*

$$f_{\psi, \phi}(a) := \langle \psi, \widetilde{K}_a(\Omega)\phi \rangle, \quad q_{\psi, \phi}(a) := \langle \psi, q(a, \Omega)\phi \rangle = -\frac{1}{2\pi} \partial_a f_{\psi, \phi}(a).$$

Assume in addition that  $q_{\psi, \phi}$  is weakly differentiable in  $a$ , that

$$\lim_{a \rightarrow +\infty} f_{\psi, \phi}(a) = 0, \quad \lim_{a \rightarrow +\infty} q_{\psi, \phi}(a) = 0,$$

and define distributionally

$$\langle \psi, p(a, \Omega)\phi \rangle := -\partial_a q_{\psi, \phi}(a) = \frac{1}{2\pi} \partial_a^2 f_{\psi, \phi}(a).$$

Then

$$q_{\psi,\phi}(a) = \int_a^\infty \langle \psi, p(v, \Omega) \phi \rangle dv,$$

and

$$\langle \psi, \widetilde{K}_a(\Omega) \phi \rangle = 2\pi \int_a^\infty (v - a) \langle \psi, p(v, \Omega) \phi \rangle dv.$$

This statement is not part of the present fixed-cutoff bridge; it records the exact additional input/output package carried into the downstream density-upgrade task.

**Null half-line blow-up net.** Fix a smooth cut point and choose affine coordinate  $v$  on generator  $\Omega$  so that the cut sits at  $v = 0$ . For  $a \geq 0$ , write

$$H_a := (a, \infty), \quad \mathcal{M}_a(\Omega) := \overline{\bigvee_{a < c < d < \infty} \mathcal{A}((c, d), \Omega)}.$$

Thus  $\mathcal{M}_a(\Omega)$  is the half-line blow-up algebra generated by bounded null interval algebras inside  $H_a$ . By isotony of the null interval net,

$$a \leq b \implies \mathcal{M}_b(\Omega) \subseteq \mathcal{M}_a(\Omega).$$

**Corollary 6.11** (Derived half-sided modular inclusion on null half-lines). *On the OPH geometric scaling branch of Theorem 6.3, the blow-up modular action near a smooth entangling cut acts on the null coordinate by*

$$v \mapsto e^{-2\pi t} v.$$

Therefore, for every  $a \geq 0$ ,

$$\sigma_t^\omega(\mathcal{M}_a(\Omega)) = \mathcal{M}_{e^{-2\pi t} a}(\Omega).$$

Hence, for every  $a > 0$ ,

$$\sigma_t^\omega(\mathcal{M}_a(\Omega)) \subseteq \mathcal{M}_a(\Omega) \quad (t \leq 0),$$

so the inclusion

$$\mathcal{M}_a(\Omega) \subset \mathcal{M}_0(\Omega)$$

is half-sided modular. Equivalently, after the harmless convention change  $t \mapsto -t$ , one recovers the standard positive-time half-sided-inclusion form. The former half-sided-inclusion premise is therefore derived from OPH null-interval structure, isotony, and the scaling-limit geometric action rather than imported as an additional modular-QFT ingredient.

*Proof.* By Theorem 6.3, on every controlled collar family whose carried remainder vanishes in the refinement limit, the cap modular flow converges to the exact geometric cap dilation with  $2\pi$  normalization. Blow up the cap near a smooth cut and restrict to the chosen null generator. In that tangent limit, the cap-preserving flow becomes the null dilation  $v \mapsto e^{-2\pi t} v$ . For every bounded interval  $(c, d) \subset H_a$ , the blow-up action therefore sends

$$\mathcal{A}((c, d), \Omega) \mapsto \mathcal{A}((e^{-2\pi t} c, e^{-2\pi t} d), \Omega).$$

Taking the von Neumann closure of the algebra generated by all such intervals gives

$$\sigma_t^\omega(\mathcal{M}_a(\Omega)) = \mathcal{M}_{e^{-2\pi t} a}(\Omega).$$

If  $t \leq 0$ , then  $e^{-2\pi t} a \geq a$ , hence

$$H_{e^{-2\pi t} a} \subseteq H_a.$$

Isotony of the null interval net therefore implies

$$\mathcal{M}_{e^{-2\pi t}a}(\Omega) \subseteq \mathcal{M}_a(\Omega).$$

This is exactly the half-sided modular-inclusion property for the pair  $\mathcal{M}_a(\Omega) \subset \mathcal{M}_0(\Omega)$ . Reparametrizing the modular parameter by  $t \mapsto -t$  gives the more common positive-time convention if desired.  $\square$

**Lemma 6.12** (Positive null-translation generator). *For the derived half-sided modular inclusion*

$$\mathcal{M}_a(\Omega) \subset \mathcal{M}_0(\Omega) \quad (a > 0)$$

*of Corollary 6.11, let  $\Delta_0(\Omega)$  be the modular operator of the standard pair  $(\mathcal{M}_0(\Omega), \omega)$  and define*

$$K_0(\Omega) := -\log \Delta_0(\Omega).$$

*Then Borchers–Wiesbrock yields a unique strongly continuous one-parameter unitary group*

$$U_\Omega(a) = e^{iaP_\Omega}, \quad a \in \mathbb{R},$$

*such that*

$$U_\Omega(a)\omega = \omega, \quad U_\Omega(a)\mathcal{M}_b(\Omega)U_\Omega(a)^* = \mathcal{M}_{a+b}(\Omega) \quad (a, b \geq 0),$$

*and whose generator  $P_\Omega$  is positive and self-adjoint on the Stone domain*

$$D(P_\Omega) := \left\{ \psi \in \mathcal{H} : \lim_{a \rightarrow 0} \frac{U_\Omega(a)\psi - \psi}{ia} \text{ exists} \right\}.$$

*Moreover,*

$$\Delta_0(\Omega)^{it}U_\Omega(a)\Delta_0(\Omega)^{-it} = U_\Omega(e^{-2\pi t}a), \quad \Delta_0(\Omega)^{it}P_\Omega\Delta_0(\Omega)^{-it} = e^{-2\pi t}P_\Omega,$$

*and on the common invariant analytic core of  $K_0(\Omega)$  and  $P_\Omega$ ,*

$$[K_0(\Omega), P_\Omega] = -i2\pi P_\Omega.$$

*For each  $a \geq 0$ , let  $\Delta_a(\Omega)$  be the modular operator of the translated standard pair  $(\mathcal{M}_a(\Omega), \omega)$  and define*

$$K_a(\Omega) := -\log \Delta_a(\Omega).$$

*Then*

$$\Delta_a(\Omega) = U_\Omega(a)\Delta_0(\Omega)U_\Omega(a)^*, \quad K_a(\Omega) = U_\Omega(a)K_0(\Omega)U_\Omega(a)^*,$$

*hence*

$$K_a(\Omega) = K_0(\Omega) - 2\pi a P_\Omega$$

*as a quadratic-form identity on  $D(K_0(\Omega)) \cap D(P_\Omega)$ . Equivalently,*

$$\langle \psi, (K_b(\Omega) - K_a(\Omega))\phi \rangle = -2\pi(b-a) \langle \psi, P_\Omega\phi \rangle$$

*for all  $\psi, \phi \in D(K_0(\Omega)) \cap D(P_\Omega)$ .*

*In the canonical blockwise normalization of Corollary 6.7, where the half-line endpoint term has been absorbed into the central part, the weak endpoint derivative of Proposition 6.9 agrees with the Borchers generator:*

$$\langle \psi, P_\Omega\phi \rangle = -\frac{1}{2\pi} \frac{d}{da} \Big|_{a=0^+} \langle \psi, \widetilde{K}_a(\Omega)\phi \rangle$$

*for the same domain vectors. The half-sided-inclusion step proves only this affine half-line pair  $(K_0(\Omega), P_\Omega)$ ; bounded-interval modular-Hamiltonian formulas remain downstream consequences of the separate interval-preserving projective branch recorded in Theorem 6.13.*

*Proof.* Corollary 6.11 gives the required half-sided modular inclusion. The standard Borchers–Wiesbrock theorem for a standard half-sided inclusion [6] then yields a unique strongly continuous unitary group  $U_\Omega(a)$  with  $U_\Omega(a)\omega = \omega$ , the translation law for the nested half-line net, and a positive self-adjoint generator  $P_\Omega$ ; Stone’s theorem gives the displayed domain formula.

The affine covariance relation

$$\Delta_0(\Omega)^{it}U_\Omega(a)\Delta_0(\Omega)^{-it} = U_\Omega(e^{-2\pi t}a)$$

is the Borchers relation. Differentiating at  $a = 0$  gives

$$\Delta_0(\Omega)^{it}P_\Omega\Delta_0(\Omega)^{-it} = e^{-2\pi t}P_\Omega.$$

Since  $\Delta_0(\Omega)^{it} = e^{-itK_0(\Omega)}$ , differentiating at  $t = 0$  on the common analytic core yields

$$-i[K_0(\Omega), P_\Omega] = -2\pi P_\Omega,$$

hence

$$[K_0(\Omega), P_\Omega] = -i2\pi P_\Omega.$$

Because  $U_\Omega(a)\omega = \omega$  and

$$\mathcal{M}_a(\Omega) = U_\Omega(a)\mathcal{M}_0(\Omega)U_\Omega(a)^*,$$

the modular operator of the translated standard pair is

$$\Delta_a(\Omega) = U_\Omega(a)\Delta_0(\Omega)U_\Omega(a)^*,$$

so

$$K_a(\Omega) = U_\Omega(a)K_0(\Omega)U_\Omega(a)^*.$$

Differentiating this identity in  $a$  on  $D(K_0(\Omega)) \cap D(P_\Omega)$  gives

$$\frac{d}{da}K_a(\Omega) = i[P_\Omega, K_a(\Omega)] = -2\pi P_\Omega,$$

because the commutator relation just proved is invariant under conjugation by  $U_\Omega(a)$ . Integrating from 0 to  $a$  yields the displayed quadratic-form identity, and taking matrix elements gives the equivalent form. Proposition 6.9 had produced the weak endpoint derivative of the renormalized half-line family; in the canonical normalization of Corollary 6.7 the central term has been removed, so that weak derivative is exactly the Borchers generator. The final sentence is the explicit claim boundary: bounded intervals require the additional projective interval action not supplied by half-sided inclusion alone.  $\square$

**Theorem 6.13** (The OPH half-line generator is the local null-stress charge). *Fix a null generator  $\Omega$  and work in the canonical blockwise normalization of Corollary 6.7. Let  $\widetilde{K}_a(\Omega)$  denote the renormalized modular Hamiltonian of the half-line  $H_a$ , and let  $P_\Omega$  be the positive Borchers generator of Lemma 6.12. Then on*

$$D_\Omega := D(K_0(\Omega)) \cap D(P_\Omega)$$

one has

$$\langle \psi, P_\Omega \phi \rangle = -\frac{1}{2\pi} \frac{d}{da} \Big|_{a=0^+} \langle \psi, \widetilde{K}_a(\Omega) \phi \rangle \quad (\psi, \phi \in D_\Omega),$$

and this operator is exactly the local null-stress charge of the effective spacetime description on that same half-line family.

Equivalently, if the effective spacetime description writes the same renormalized family as

$$\widetilde{K}_a^{\text{eff}}(\Omega) = 2\pi Q_a^{\text{mod}}(\Omega) + K_\partial(a, \Omega),$$

with  $Q_a^{\text{mod}}(\Omega)$  the standard local null modular-energy term and  $K_\partial(a, \Omega)$  central and endpoint-supported, then

$$Q_{kk}(\Omega) := -\frac{1}{2\pi} \frac{d}{da} \Big|_{a=0^+} \widetilde{K}_a^{\text{eff}}(\Omega)$$

satisfies

$$P_\Omega = Q_{kk}(\Omega)$$

as a quadratic-form identity on  $D_\Omega$ .

Hence the renormalized OPH null generator is not a separate EFT-side premise: it is the same operator that the effective spacetime description calls the local null-stress charge of the null half-line.

*Proof.* Lemma 6.12 proves the first displayed identity: in the canonical normalization of Corollary 6.7, the weak endpoint derivative of the renormalized half-line family is exactly the Borchers generator. Passing to the effective spacetime description does not change the operator family; it rewrites the same  $\widetilde{K}_a(\Omega)$  in local continuum variables appropriate to the local Lorentzian regime. In that description, the local null-stress charge is precisely the right endpoint derivative

$$-\frac{1}{2\pi} \frac{d}{da} \Big|_{a=0^+} \widetilde{K}_a^{\text{eff}}(\Omega)$$

of the same half-line modular family. Because  $\widetilde{K}_a^{\text{eff}}(\Omega)$  and  $\widetilde{K}_a(\Omega)$  represent the same renormalized modular Hamiltonians, their right derivatives agree on  $D_\Omega$ . Therefore the OPH generator and the effective null-stress charge coincide:

$$P_\Omega = Q_{kk}(\Omega).$$

The endpoint-supported central term does not alter the noncentral generator; in the canonical blockwise normalization it is absorbed into the central part fixed in Corollary 6.7. This proves the identification.  $\square$

**Imported input and remaining boundary.** The only imported continuum input used above is the standard local null modular-Hamiltonian formula for the effective spacetime description of the same half-line family. No separate “null-stress identification premise” remains. What is still downstream is only:

- (i) transport of the half-line statement to bounded null intervals, which still requires the separate interval-preserving projective branch and its affine-covariant kernel  $g_I(v)$ ; and
- (ii) reconstruction of a full local symmetric tensor from the directional charges, which is determined only up to the null-invisible metric term  $\phi g_{ab}$ .

So the gap closed here is the generator/charge identification itself: inside the null bridge and on the declared half-line family,  $P_\Omega$  is exactly the local null-stress charge.

**Lemma 6.14** (Null data determine the stress tensor modulo a metric term). *Let  $X_{ab}$  be a symmetric tensor. If*

$$X_{ab} k^a k^b = 0$$

*for every null vector  $k$  at a point, then*

$$X_{ab} = \phi g_{ab}$$

*for some scalar  $\phi$ .*

*Proof.* Write

$$X_{ab} = Y_{ab} + \phi g_{ab}$$

with  $Y_{ab}$  traceless. Since  $g_{ab}k^ak^b = 0$  on null vectors, the hypothesis implies  $Y_{ab}k^ak^b = 0$  for all null  $k$ . In local inertial coordinates let  $k = (1, \hat{n})$  with  $|\hat{n}| = 1$ . Then

$$Y_{ab}k^ak^b = Y_{00} + 2\hat{n}^i Y_{0i} + \hat{n}^i \hat{n}^j Y_{ij}.$$

Oddness under  $\hat{n} \mapsto -\hat{n}$  forces  $Y_{0i} = 0$ . Averaging the remaining relation over  $S^2$  gives

$$Y_{00} + \frac{1}{3} \delta^{ij} Y_{ij} = 0.$$

Tracelessness of  $Y_{ab}$  gives

$$-Y_{00} + \delta^{ij} Y_{ij} = 0,$$

so both  $Y_{00}$  and the spatial trace  $\delta^{ij} Y_{ij}$  vanish. The remaining condition is therefore

$$\hat{n}^i \hat{n}^j Y_{ij}^{\text{TF}} = 0 \quad \text{for all } \hat{n} \in S^2,$$

where  $Y_{ij}^{\text{TF}}$  is the traceless spatial part. The  $\ell = 2$  spherical harmonics are linearly independent, so this implies  $Y_{ij}^{\text{TF}} = 0$ . Hence  $Y_{ab} = 0$  and therefore  $X_{ab} = \phi g_{ab}$ .  $\square$

**Remark 6.15** (Claim boundary in the null bridge). *The c.12-local bridge ends with explicit theorem-level objects: the positive self-adjoint null-translation generator  $P_\Omega$  on its Stone domain, its Borchers affine covariance relation, the half-line modular identities above, and the exact half-line generator/charge identification of Theorem 6.13. What remains downstream is only transport to bounded null intervals through the separate interval-preserving projective branch and the later tensor upgrade modulo the null-invisible metric term. The density-upgrade template of Theorem 6.10 is recorded separately and does not carry the operator-identification burden.*

### 6.3 Generalized entropy and the Einstein equation

From this point through Corollary 6.20 we specialize to the physical  $d = 4$  scaling branch. The claim boundary is explicit. The only standard geometric input used below, beyond the OPH outputs D3 and D4, is the fixed-volume area-variation identity for a small geodesic ball. The small-ball modular kernel used below is *not* imported from a separate EFT law: it is the local expression of the OPH geometric cap generator  $B_C$  from Theorem 6.3, evaluated using the half-line null-stress identification of Theorem 6.13 together with the separate interval-preserving projective branch that transports it to bounded null intervals.

For a reference state  $\omega$  and a small cap  $C$ ,

$$\delta S_C = \delta \langle K_C \rangle.$$

By Theorem 6.3,

$$\delta S_C = 2\pi \delta \langle B_C \rangle.$$

By Proposition 5.16,

$$S_{\text{gen}}(C) = \text{Tr}(\rho L_C) + S_{\text{bulk}}(C).$$

Together with Definition 5.17, this identifies the area term as the coarse-grained form of the edge entropy density rather than an independent postulate. After this edge-center split, the first-law piece entering the small-ball argument is the bulk contribution:

$$\delta S_{\text{bulk}}(C) = 2\pi \delta \langle B_C \rangle.$$

**Definition 6.16** (Admissible equilibrium variation). *Throughout this subsection,  $\delta$  denotes a first-order state variation about a reference state  $\omega$ , within the cap-label-preserving MaxEnt class, at fixed cap  $C$ , fixed cap size/volume, and fixed conserved charges. Shape or null-cut deformations are denoted separately by  $\delta_{\text{shape}}$  or  $\partial_\lambda$ .*

**Assumption 6.17** (Fixed-cap generalized-entropy stationarity). *For admissible variations preserving cap labels, cap size/volume, and relevant conserved charges, the realized reference state is stationary for generalized entropy:*

$$\delta S_{\text{gen}}(C) = 0.$$

This is the Jacobson-type fixed-cap stationarity premise used in this paper; it is not derived from MaxEnt alone.

**Lemma 6.18** (Internal  $d=4$  small-ball bridge). *Assume the hypotheses of Theorem 6.13, together with the separate interval-preserving projective branch invoked there for bounded intervals. Let  $p$  be the cap center and let  $u^a$  be the future-directed unit tangent of the local diamond rest frame at  $p$ . In local inertial coordinates  $(t, x^i)$  adapted to  $u^a$ , let  $D_\ell$  be the causal diamond whose  $t = 0$  slice is the Euclidean ball*

$$B_\ell = \{t = 0, r < \ell\}, \quad r^2 = \delta_{ij}x^i x^j.$$

Then

$$\delta S_{\text{bulk}}(C) = 2\pi \int_{B_\ell} \frac{\ell^2 - r^2}{2\ell} \delta \langle T_{00} \rangle d^3x + \delta \langle E_{C,\ell}^{(\eta)} \rangle.$$

If, moreover,  $\delta \langle T_{00} \rangle$  is approximately constant across  $B_\ell$  and

$$\delta \langle E_{C,\ell}^{(\eta)} \rangle = o(\ell^4)$$

along the same scaling family, then in  $d = 4$ ,

$$\delta S_{\text{bulk}}(C) = \frac{8\pi^2 \ell^4}{15} \delta \langle T_{00} \rangle + O(\ell^5 \partial T) + o(\ell^4).$$

*Proof.* Work in the local Lorentzian scaling regime around  $p$ . On the geometric-subnet branch of Theorem 6.3, the cap modular flow is the geometric flow of the diamond-preserving conformal Killing field. In the tangent diamond  $D_\ell$ , that field is

$$\xi_{D_\ell} = \frac{1}{2\ell} \left( (\ell^2 - r^2 - t^2) \partial_t - 2t x^i \partial_i \right),$$

so on the  $t = 0$  slice one has

$$\xi_{D_\ell} \cdot n = \frac{\ell^2 - r^2}{2\ell},$$

with  $n^a = u^a$  the slice normal at the center.

The same kernel is the bounded-interval kernel supplied by that separate projective branch. Along each null generator of the diamond, Theorem 6.13 identifies the OPH half-line generator with the local null-stress charge, and the interval-preserving projective branch transports that identification to the affine-covariant interval weight that vanishes at the two diamond endpoints. On the  $t = 0$  slice that weight is exactly  $(\ell^2 - r^2)/(2\ell)$ . Therefore the null-stress bridge reconstructs the bulk modular charge of the geometric generator as

$$\delta \langle B_C \rangle = \int_{B_\ell} \frac{\ell^2 - r^2}{2\ell} \delta \langle T_{00} \rangle d^3x + \frac{1}{2\pi} \delta \langle E_{C,\ell}^{(\eta)} \rangle.$$

Multiplying by the first-law factor  $2\pi$  gives the displayed formula for  $\delta S_{\text{bulk}}(C)$ .

If  $\delta\langle T_{00}\rangle$  is approximately constant across  $B_\ell$ , then

$$\int_{B_\ell} \frac{\ell^2 - r^2}{2\ell} \delta\langle T_{00}\rangle d^3x = \frac{4\pi\ell^4}{15} \delta\langle T_{00}\rangle + O(\ell^5\partial T).$$

The additional hypothesis  $\delta\langle E_{C,\ell}^{(\eta)}\rangle = o(\ell^4)$  then yields

$$\delta S_{\text{bulk}}(C) = 2\pi \cdot \frac{4\pi\ell^4}{15} \delta\langle T_{00}\rangle + O(\ell^5\partial T) + o(\ell^4),$$

which is exactly the stated coefficient. No separate EFT small-ball first law has been used: the kernel comes from the OPH geometric generator together with the D4 null-stress bridge.  $\square$

**Theorem 6.19** (Conditional Jacobson-type rest-frame relation from entanglement equilibrium). *Assume Axioms 3.1–3.4, Assumption 3.6, Assumption 6.17, and the hypotheses of Theorem 6.13 and Lemma 6.18. Then, for every sufficiently small cap centered at  $p$  and every admissible first variation  $\delta$  about a reference state  $\omega$  in the locally Lorentzian  $d = 4$  scaling regime, if  $u^a$  is the future-directed unit tangent of the local diamond rest frame at  $p$ , one has*

$$\delta[(G_{ab} + \Lambda g_{ab})u^a u^b] = 8\pi G u^a u^b \delta\langle T_{ab}\rangle$$

at  $p$ . In the adapted rest frame  $u^a = (1, 0, 0, 0)$ , this is

$$\delta(G_{00} + \Lambda g_{00}) = 8\pi G \delta\langle T_{00}\rangle.$$

*Proof.* Assumption 6.17 gives the fixed-cap stationarity condition

$$0 = \delta S_{\text{gen}}(C) = \delta S_{\text{bulk}}(C) + \frac{\delta A}{4G}.$$

By Lemma 6.18,

$$\delta S_{\text{bulk}}(C) = \frac{8\pi^2\ell^4}{15} u^a u^b \delta\langle T_{ab}\rangle + O(\ell^5\partial T) + o(\ell^4),$$

where in the adapted rest frame the contraction is  $\delta\langle T_{00}\rangle$ .

For a fixed-volume small geodesic ball in  $d = 4$ , the standard area-variation identity gives

$$\delta A|_{V,\Lambda} = -\frac{4\pi\ell^4}{15} \delta[(G_{ab} + \Lambda g_{ab})u^a u^b].$$

Therefore

$$0 = \frac{8\pi^2\ell^4}{15} u^a u^b \delta\langle T_{ab}\rangle - \frac{\pi\ell^4}{15G} \delta[(G_{ab} + \Lambda g_{ab})u^a u^b] + O(\ell^5\partial T) + o(\ell^4).$$

Dividing by  $\ell^4$  and taking the small-ball limit removes the  $O(\ell^5\partial T) + o(\ell^4)$  terms and yields

$$\delta[(G_{ab} + \Lambda g_{ab})u^a u^b] = 8\pi G u^a u^b \delta\langle T_{ab}\rangle.$$

This is precisely the desired rest-frame scalar relation.  $\square$

**Corollary 6.20** (Internal tensor upgrade in the scaling regime). *If the first-variation relation of Theorem 6.19 holds for all local observer four-velocities and all reference states in a connected scaling branch, then*

$$G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab}\rangle$$

throughout that branch. The only local ambiguity is the metric term isolated by Lemma 6.14, and that ambiguity is exactly the same branch constant  $\Lambda$ .

*Proof.* Define

$$Y_{ab} := G_{ab} + \Lambda g_{ab} - 8\pi G \langle T_{ab} \rangle.$$

At any point  $p$ , Theorem 6.19 gives

$$u^a u^b \delta Y_{ab} = 0$$

for the unit future-directed four-velocity  $u^a$  of the local diamond rest frame. By the Lorentz branch of Theorem 6.3 and overlap consistency across observers through the same point, these local rest frames exhaust all timelike directions. Hence

$$u^a u^b \delta Y_{ab} = 0 \quad \text{for every unit timelike } u^a.$$

Choose local inertial coordinates at  $p$  and write

$$u^a = \gamma(1, v^i), \quad |\vec{v}| < 1, \quad \gamma = (1 - |\vec{v}|^2)^{-1/2}.$$

Then

$$0 = u^a u^b \delta Y_{ab} = \gamma^2 \left( \delta Y_{00} + 2v^i \delta Y_{0i} + v^i v^j \delta Y_{ij} \right)$$

for every  $|\vec{v}| < 1$ . The quadratic polynomial in  $v^i$  therefore vanishes on an open ball, so all of its coefficients vanish:

$$\delta Y_{00} = 0, \quad \delta Y_{0i} = 0, \quad \delta Y_{ij} = 0.$$

Thus

$$\delta Y_{ab} = 0$$

as a full tensor.

Along any path of admissible first variations inside the connected scaling branch,  $Y_{ab}$  is therefore constant. The null bridge isolated the only purely local freedom as a metric term via Lemma 6.14; fixing that term on one maximally symmetric reference state determines the same branch constant  $\Lambda$ . Equivalently,

$$Y_{ab} = 0,$$

so

$$G_{ab} + \Lambda g_{ab} = 8\pi G \langle T_{ab} \rangle$$

throughout the branch. □

## 6.4 The cosmological constant

**Lemma 6.21** (Vacuum-Energy Blindness). *For any null vector  $k$  and any vacuum-energy contribution  $T_{ab}^{\text{vac}} = -\rho_{\text{vac}} g_{ab}$ ,*

$$T_{kk}^{\text{vac}} = 0.$$

**Proposition 6.22** (Structural Separation of  $\Lambda$ ). *Local null-modular data fix  $T_{ab}$  only up to  $\phi g_{ab}$ . Therefore  $\Lambda$  is not determined by local overlap consistency and requires a global input.*

**Corollary 6.23** (Cosmological Constant from Capacity). *If the global screen capacity is identified with the de Sitter static-patch entropy,*

$$N_{\text{scr}} = S_{\text{dS}} = \frac{A_{\text{dS}}}{4G} = \frac{3\pi}{G\Lambda},$$

then

$$\Lambda = \frac{3\pi}{GN_{\text{scr}}}.$$

Proposition 6.22 and Corollary 6.23 are the intended local/global D6 package: local null-modular reconstruction leaves the Einstein equation determined only up to a metric term, and the global screen-capacity identification fixes the remaining constant. This is a structural aspect of the framework. The vacuum-energy problem is reorganized by separating local null data from the global capacity term.

## 7 Gauge Reconstruction and Standard Model Structure

### 7.1 Compact gauge reconstruction

At any fixed UV cutoff, edge-center completion equips collars with finitely many sector labels and fusion rules. These are fixed-cutoff collar data. The local MaxEnt / collar-mixing package established earlier controls only fixed-cutoff recoverability, modular-support localization, and carried error terms on the realized branch. It is logically separate from the question whether transportable edge sectors survive refinement, and it supplies no theorem that the refinement-limit sector category is trivial or nontrivial. A later nontrivial transportable colimit is therefore treated here as compatible additional gauge data rather than as something decided by the mixing estimate itself. Whenever compact gauge reconstruction is invoked, the entire transportable refinement-directed sector system is part of Assumption 3.7.

Assume therefore that in the EFT regime there is a directed system  $(\text{Sect}_r)_r$  of transportable edge-sector categories with refinement functors, and write

$$\text{Sect}_\infty := \varinjlim_r \text{Sect}_r$$

for the directed colimit retaining the transportable sectors and intertwiners that persist in that system, with objectwise finite-dimensional fibers. This is the category on which Doplicher–Roberts / Tannaka reconstruction is applied. The theorem below is neutral about whether  $\text{Sect}_\infty$  is trivial or nontrivial: if the transportable colimit of Assumption 3.7 furnished only the tensor unit, the reconstructed compact group would be the trivial group.

**Theorem 7.1** (Conditional compact gauge reconstruction in the bosonic branch). *Assume Axioms 3.1–3.4 and Assumption 3.7. Let*

$$\mathcal{F} : \text{Sect}_\infty \rightarrow \text{Hilb}_{\text{fd}}$$

*be the faithful bosonic fiber functor on that transportable colimit in the EFT regime. If  $\text{Sect}_\infty$  is a rigid symmetric  $C^*$ -tensor category, then*

$$G := \text{Aut}_\otimes(\mathcal{F})$$

*is a compact group and*

$$\text{Sect}_\infty \simeq \text{Rep}(G).$$

*In particular  $G$  is uniquely determined up to isomorphism.*

*Proof.* The theorem begins only once the directed system of transportable sectors and its objectwise finite-dimensional fiber functor are given; those data are part of Assumption 3.7, not consequences of the local-Gibbs or Dobrushin-type collar estimates. Fix a small skeleton of  $\text{Sect}_\infty$ , which does not change the tensor category or the fiber functor up to equivalence. For every object  $X$ , any

monoidal natural automorphism  $\eta \in \text{Aut}_\otimes(\mathcal{F})$  has a unitary component  $\eta_X \in U(\mathcal{F}(X))$  because  $\mathcal{F}$  is a  $*$ -functor to finite-dimensional Hilbert spaces. Hence there is an injective map

$$\text{Aut}_\otimes(\mathcal{F}) \hookrightarrow \prod_X U(\mathcal{F}(X)), \quad \eta \mapsto (\eta_X)_X.$$

For each intertwiner  $f : X \rightarrow Y$ , naturality imposes the closed relation  $\mathcal{F}(f)\eta_X = \eta_Y\mathcal{F}(f)$ . For each pair  $X, Y$ , the monoidal structure isomorphism  $J_{X,Y} : \mathcal{F}(X) \otimes \mathcal{F}(Y) \xrightarrow{\sim} \mathcal{F}(X \otimes Y)$  imposes the closed relation

$$\eta_{X \otimes Y} = J_{X,Y} (\eta_X \otimes \eta_Y) J_{X,Y}^{-1}, \quad \eta_{\mathbf{1}} = \text{id}_{\mathbb{C}}.$$

Therefore  $\text{Aut}_\otimes(\mathcal{F})$  is a closed subgroup of the compact product  $\prod_X U(\mathcal{F}(X))$ , so  $G = \text{Aut}_\otimes(\mathcal{F})$  is compact.

The remaining hypotheses required by Doplicher–Roberts / Tannaka reconstruction are now exactly checked on the declared sector category  $\text{Sect}_\infty$ :  $\text{Sect}_\infty$  is rigid, symmetric, and  $C^*$  by assumption, and  $\mathcal{F}$  is a faithful bosonic fiber functor with finite-dimensional values by Assumption 3.7. The standard reconstruction theorem therefore applies to the pair  $(\text{Sect}_\infty, \mathcal{F})$  and yields a symmetric  $C^*$ -tensor equivalence

$$\text{Sect}_\infty \simeq \text{Rep}(G),$$

where an object  $X$  is sent to the representation  $g \mapsto g_X$  on  $\mathcal{F}(X)$  and morphisms are sent to the corresponding intertwiners [16, 17]. If the same category were also equivalent to  $\text{Rep}(G')$  through another faithful bosonic fiber functor, then composing the two reconstruction equivalences would give a symmetric tensor equivalence  $\text{Rep}(G) \simeq \text{Rep}(G')$  compatible with the forgetful fiber functors. Applying the same reconstruction theorem to either side identifies the two automorphism groups of the fiber functor, so  $G \cong G'$ . Thus the reconstructed compact group is unique up to isomorphism.  $\square$

If a fermionic sign object is present, the correct reconstruction statement is super-Tannakian rather than purely Tannakian. This paper does not prove that fermionic/chiral extension; it works only in the bosonic internal-gauge branch.

## 7.2 Minimal admissibility and the Standard Model quotient

Throughout this subsection,  $\chi_{\text{cpl}}$  means the dimension of the minimal coupled carrier supporting the required weak-type and color-type nonabelian charges on a common block. It is not the abstract minimal faithful representation dimension of the final gauge group.

**Definition 7.2** (Weak-type and color-type nonabelian roles). *A weak-type nonabelian charge is a pseudoreal nonabelian doublet role. A color-type nonabelian charge is an intrinsically complex nonabelian role, meaning that its nonabelian factor itself acts by an irreducible representation not equivalent to its conjugate. Abelian twisting of a pseudoreal nonabelian irreducible representation does not count as a color-type role.*

The derivation of the Standard Model quotient separates into five steps:

1. existence of a compact reconstructed group;
2. identification of the minimal nonabelian sector content;
3. identification of the minimal coupled carrier;
4. uniqueness of the connected abelian factor once admitted on that carrier;

5. determination of product gauge structure up to finite quotient, then of the global finite quotient from the realized matter spectrum.

**Lemma 7.3** (Minimal nonabelian sector content). *Any admissible low-energy sector must contain both a weak-type pseudoreal nonabelian charge role and an intrinsically complex color-type nonabelian charge role.*

*Proof.* Light chiral matter requires left-handed multiplets and right-handed singlets to carry inequivalent gauge data. A pseudoreal doublet structure is the minimal way to realize the weak sector without immediately producing vectorlike masses. A genuinely complex nonabelian representation is required to distinguish quarks from antiquarks. A single nonabelian simple factor cannot simultaneously supply both a minimal weak-type role and an intrinsically complex color-type role; abelian twisting does not count toward the color-type requirement.  $\square$

**Lemma 7.4** (Connected 2D pseudoreal image classification). *Let  $H$  be a connected compact group with a faithful irreducible 2-dimensional pseudoreal unitary representation. Then the connected derived subgroup of its image is conjugate to  $SU(2)$ . Equivalently, the nonabelian factor acting in that role is  $SU(2)$  up to finite central quotient.*

*Proof sketch.* Irreducibility places the image inside  $U(2)$  with scalar commutant. Its connected derived subgroup is therefore a connected compact subgroup of  $SU(2)$ . The connected compact subgroups of  $SU(2)$  are either tori or  $SU(2)$  itself. The representation is nonabelian and pseudoreal, so the torus case is excluded. Hence the derived subgroup is conjugate to  $SU(2)$ , with only a finite central kernel left invisible in the representation.  $\square$

**Lemma 7.5** (Connected 3D intrinsically complex image classification). *Let  $H$  be a connected compact group with a faithful irreducible intrinsically complex 3-dimensional unitary representation. Then the connected derived subgroup of its image is conjugate to  $SU(3)$ . Equivalently, the nonabelian factor acting in that role is  $SU(3)$  up to finite central quotient.*

*Proof sketch.* Because the representation is irreducible on  $\mathbb{C}^3$ , the semisimple part of the Lie algebra of the image acts irreducibly on  $\mathbb{C}^3$ . A product of two nontrivial simple factors would force a tensor-product decomposition of dimension at least 4, so only one simple factor can occur. Among compact simple Lie algebras, the only ones with nontrivial irreducible representations of dimension at most 3 are  $\mathfrak{su}(2)$  and  $\mathfrak{su}(3)$ . The 3-dimensional  $\mathfrak{su}(2)$  representation is real, not intrinsically complex, whereas the fundamental  $\mathfrak{su}(3)$  representation is intrinsically complex. Therefore the connected derived subgroup is conjugate to  $SU(3)$ , again up to finite central kernel.  $\square$

**Lemma 7.6** (Minimal coupled carrier under MAR). *Within the connected positive-dimensional Lie admissible class used by MAR, and under the weak-type / intrinsically complex color-type criteria above, the minimal coupled carrier supporting both roles on a common block has the form*

$$V = \mathbb{C}^3 \otimes \mathbb{C}^2, \quad \chi_{\text{cpl}} = 6.$$

*Proof.* By Lemma 7.4, the minimal weak-type nonabelian role is the pseudoreal doublet of  $SU(2)$ , hence has dimension 2. By Lemma 7.5, the minimal intrinsically complex color-type nonabelian role is the triplet of  $SU(3)$ , hence has dimension 3.

By definition of  $\chi_{\text{cpl}}$ , the weak-type and color-type roles must act nontrivially on one common irreducible nonabelian block rather than on disconnected direct-sum summands. Commuting irreducible actions on such a common block therefore require dimension at least the product of the minimal weak and color dimensions:

$$\chi_{\text{cpl}} \geq 2 \cdot 3 = 6.$$

Equality is achieved by the tensor-product carrier  $\mathbb{C}^3 \otimes \mathbb{C}^2$ . The block-diagonal representation  $\mathbb{C}^3 \oplus \mathbb{C}^2$  of  $S(U(3) \times U(2))$  is faithful of dimension 5, but it is not coupled and hence is not the MAR minimizer. Connected non-semisimple counterexamples of  $U(2)$  type are excluded by the explicit color-type criterion rather than being silently ignored.  $\square$

**Remark 7.7.** *The dimension comparisons used in Lemma 7.6 apply to the positive-dimensional connected Lie image carrying the admissible nonabelian charges, equivalently to the identity component of the reconstructed group on that sector. Finite, disconnected, or connected non-semisimple counterexamples exist, but they are outside the admissible class fixed by the explicit weak-type / color-type criteria.*

MAR supplies existence of a connected abelian factor; the next lemma proves only uniqueness and identification on the minimal coupled carrier.

**Lemma 7.8** (Uniqueness of the connected abelian factor on the minimal coupled carrier). *Inside  $U(6)$ , the commutant of  $SU(3) \times SU(2)$  acting on  $\mathbb{C}^3 \otimes \mathbb{C}^2$  is exactly  $U(1)$ .*

*Proof.* The  $SU(3)$  action is irreducible on the first tensor factor and trivial on the second. The  $SU(2)$  action is irreducible on the second tensor factor and trivial on the first. By Schur's lemma, any operator commuting with both actions is scalar on each irreducible factor. Hence the full commutant is a single  $U(1)$ .  $\square$

**Theorem 7.9** (Product Gauge Structure up to Finite Quotient). *Assume Axiom 3.5, the hypotheses of Theorem 7.1 on the ordinary or central-defect realized low-energy branch, the weak-type / color-type MAR premises above, and that the realized connected gauge image acts faithfully on the minimal coupled carrier. Then the realized connected gauge structure has the form*

$$G_{\text{phys}} = \frac{SU(3) \times SU(2) \times U(1)}{\Gamma}$$

for some finite central subgroup  $\Gamma$ .

*Proof.* Theorem 7.1 yields some compact group  $G$ . Lemma 7.3 identifies the minimal admissible sector content, Lemmas 7.4 and 7.5 identify the connected nonabelian factors realizing the minimal weak-type and color-type roles, and Lemma 7.6 fixes the minimal coupled carrier. The tensor-product structure of that carrier implies commuting weak and color actions, so the connected semisimple part of the realized image is locally  $SU(3) \times SU(2)$ . Axiom 3.5 supplies the existence of one connected abelian charge factor acting nontrivially on the coupled carrier, and Lemma 7.8 shows that any such connected abelian factor is necessarily a single  $U(1)$ . A compact connected Lie group with Lie algebra  $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$  is a quotient of  $SU(3) \times SU(2) \times U(1)$  by a finite central subgroup. Faithfulness rules out an invisible extra torus, so the only remaining ambiguity is the finite central quotient  $\Gamma$ .  $\square$

**Theorem 7.10** (Hypercharge lattice on the realized matter package). *Assume gauge group  $SU(N_c) \times SU(2) \times U(1)_Y$ , one generation of chiral matter  $(Q, u^c, d^c, L, e^c)$ , one Higgs doublet  $H$ , and Yukawa terms*

$$QH u^c, \quad QH^\dagger d^c, \quad LH^\dagger e^c.$$

*Then anomaly cancellation and Yukawa invariance determine the hypercharge ratios up to an overall  $U(1)_Y$  normalization:*

$$Y_L = -N_c Y_Q, \quad Y_H = N_c Y_Q, \quad Y_u = -(N_c + 1) Y_Q, \quad Y_d = (N_c - 1) Y_Q, \quad Y_e = 2N_c Y_Q.$$

Fixing the normalization by  $Q = T_3 + Y$  and  $Q(\nu_L) = 0$  gives

$$Y_Q = \frac{1}{2N_c}.$$

For  $N_c = 3$  one recovers the exact Standard Model lattice

$$Y_Q = \frac{1}{6}, \quad Y_L = -\frac{1}{2}, \quad Y_u = -\frac{2}{3}, \quad Y_d = \frac{1}{3}, \quad Y_e = 1, \quad Y_H = \frac{1}{2}.$$

*Proof.* Yukawa invariance gives

$$Y_u = -(Y_Q + Y_H), \quad Y_d = -Y_Q + Y_H, \quad Y_e = -Y_L + Y_H.$$

The mixed anomalies yield

$$N_c Y_Q + Y_L = 0,$$

and

$$2N_c Y_Q + N_c Y_u + N_c Y_d + 2Y_L + Y_e = 0.$$

Solving the first anomaly gives

$$Y_L = -N_c Y_Q.$$

Substituting the Yukawa relations and  $Y_L = -N_c Y_Q$  into the mixed gravitational anomaly then yields

$$Y_H = N_c Y_Q, \quad Y_u = -(N_c + 1)Y_Q, \quad Y_d = (N_c - 1)Y_Q, \quad Y_e = 2N_c Y_Q.$$

With these relations in place, the  $SU(N_c)^2 U(1)$  and  $U(1)^3$  anomalies cancel identically. The normalization of  $Y_Q$  is fixed by the electric charge operator  $Q = T_3 + Y$  together with  $Q(\nu_L) = 0$ , which implies

$$Y_L = -\frac{1}{2} = -N_c Y_Q, \quad \text{hence} \quad Y_Q = \frac{1}{2N_c}.$$

□

**Corollary 7.11** (Three Generations on the realized MAR-admissible branch). *Under the hypotheses of Theorem 7.9, the asymptotic-freedom/CP-capability window together with the realized MAR selector gives*

$$N_g = 3.$$

*Proof.* Intrinsic CKM CP violation requires

$$\frac{(N_g - 1)(N_g - 2)}{2} > 0,$$

hence  $N_g \geq 3$ . For one Higgs doublet, the one-loop weak-sector coefficient is

$$b_2 = \frac{11}{3}C_A - \frac{2}{3}\sum_f T(R_f) - \frac{1}{3}\sum_s T(R_s) = \frac{22}{3} - \frac{N_g(N_c + 1)}{3} - \frac{1}{6} = \frac{43}{6} - \frac{N_g(N_c + 1)}{3}.$$

Asymptotic freedom therefore implies

$$N_g(N_c + 1) < \frac{43}{2}.$$

Since the color-type requirement forces  $N_c \geq 3$ , this gives  $N_g \leq 5$ . The realized MAR selector then chooses  $N_g = 3$ . □

**Corollary 7.12** (Three Colors on the realized MAR-admissible branch). *Under the hypotheses of Theorem 7.9, with  $N_g = 3$  from Corollary 7.11, Witten’s global SU(2) anomaly together with the realized MAR selector gives*

$$N_c = 3.$$

*Proof.* Witten’s global SU(2) anomaly requires the total number of left-handed weak doublets to be even:

$$N_g(N_c + 1) \equiv 0 \pmod{2}.$$

With  $N_g = 3$ , this implies  $N_c$  is odd. The case  $N_c = 1$  is inadmissible because it removes the intrinsically complex color sector demanded by Lemma 7.3. Minimal admissibility therefore selects  $N_c = 3$  [22].  $\square$

**Proposition 7.13** (Global quotient from the realized matter spectrum). *If the realized matter spectrum carries the hypercharges of Theorem 7.10 with  $N_c = 3$ , then the subgroup of  $SU(3) \times SU(2) \times U(1)$  acting trivially on all realized states is exactly  $\mathbb{Z}_6$ .*

*Proof.* Write  $q := 6Y \in \mathbb{Z}$  and  $\eta := e^{i\pi/3}$ , so the U(1) factor acts by  $\eta^q$  on charge- $q$  multiplets. The element

$$g_0 := (\omega_3, -1, \eta)$$

acts trivially on the realized matter and Higgs multiplets:

$$\begin{aligned} Q : \omega_3(-1)\eta &= 1, & u^c : \omega_3^{-1}\eta^{-4} &= 1, & d^c : \omega_3^{-1}\eta^2 &= 1, \\ L : (-1)\eta^{-3} &= 1, & e^c : \eta^6 &= 1, & H : (-1)\eta^3 &= 1. \end{aligned}$$

Its powers therefore generate a subgroup of order six acting trivially on all realized states. Conversely, any central element acting trivially on  $e^c$  must have U(1) part  $\eta^n$ , and triviality on  $Q$  then fixes the SU(3) and SU(2) center factors uniquely. Hence every trivial central element is a power of  $g_0$ , so the kernel is exactly  $\mathbb{Z}_6$ .  $\square$

**Corollary 7.14** (Standard Model Gauge Group). *Under Theorem 7.9, Theorem 7.10, Corollaries 7.11 and 7.12, and Proposition 7.13,*

$$G_{\text{phys}} = \frac{SU(3) \times SU(2) \times U(1)}{\mathbb{Z}_6}.$$

*Proof.* Theorem 7.9 yields the product structure up to finite quotient, and Proposition 7.13 fixes that quotient to  $\mathbb{Z}_6$  once the realized hypercharge lattice and  $N_c = 3$  are in place.  $\square$

**Corollary 7.15** (Product-Group Consequences). *The adjoint representation of the full connected gauge group contains only*

$$(8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0).$$

*Equivalently, the adjoint representation of the derived gauge group contains only*

$$(8, 1, 0) \oplus (1, 3, 0).$$

*There are no mixed  $(3, 2, \pm 5/6)$  gauge bosons. Hence gauge-mediated proton decay is absent. The unbroken gauge symmetries forbid elementary gauge-boson mass terms: the photon remains massless, while gluons remain massless gauge fields of the unbroken color symmetry.*

**Remark 7.16.** *The same product-group structure removes the usual grand-unification monopole sector. In the present framework, unification of couplings does not proceed by embedding the gauge group into a simple group that is later broken. It proceeds through a shared geometric origin of the edge heat-kernel parameters. Absence of gauge-mediated proton decay and unification therefore coexist without tension. Any confinement statement is a separate infrared dynamical issue rather than a corollary of the product-group adjoint argument alone.*

### 7.3 Fundamental scales

The quantitative D10 branch is a forward calibration sector, not part of the recovered-core theorem package. Its external continuous particle-physics input is

$$P \equiv a_{\text{cell}}/\ell_P^2,$$

together with the realized discrete gauge data of the structural branch, in particular

$$N_c = 3, \quad \beta_{\text{EW}} = N_c + 1 = 4.$$

The logical direction is therefore

$$P \mapsto (M_U(P), E_{\text{cell}}(P)) \mapsto \alpha_U(P) \mapsto (t_U(P), t_{\text{tr}}(P)) \mapsto (t_2(P), t_3(P), v(P)) \mapsto \alpha_i(\mu_*, P).$$

The edge-law input on this lane is split explicitly by claim tier. The fixed-cutoff same-overlap thermal/Casimir theorem is carried on the separate microphysics surface. This compact paper imports that closure only through the declared merge boundary: one fixed-cutoff edge-law package below, then the compact-group / Peter–Weyl lift used in the D10 pixel constraint above it. The remaining open burden is refinement-stable universality/uniqueness of that lift and the scalar-root control of the printed D10 solve, not the existence of the finite-cutoff Casimir branch itself. Measured electroweak quantities are not used to infer the internal transmutation parameters. They enter only at the end as compare-only validation targets for the printed D10 running/matching package. The forward transmutation certificate on the live D10 lane makes that logical order explicit: the source-only basis reconstructs the same  $\alpha_U(P)$ , hence the same unified diffusion parameter

$$t_U(P) = 4\pi^2 \alpha_U(P),$$

and transmutation exponent

$$t_{\text{tr}}(P) = \frac{2\pi}{(N_c + 1)\alpha_U(P)}$$

as the pixel-closure solve itself. Here the paper-side transmutation factor is the same  $\beta_{\text{EW}} = N_c + 1$  used below; older overloaded  $\beta$ -ratios are compare-only diagnostic readouts and are not part of the theorem contract.

The scales fixed directly from  $P$  are

$$M_U(P) = \frac{E_P}{e^{2\pi}} P^{1/6}, \quad E_{\text{cell}}(P) = \frac{E_P}{\sqrt{P}}.$$

The factor  $e^{-2\pi}$  is the same modular normalization used on the Lorentz/BW side, and  $P^{1/6}$  is the cell-area scaling relation carried by the present D10 implementation.

The one-dimensional internal variable solved on the D10 branch is the unified coupling  $\alpha_U$ . For a trial value of  $\alpha_U$ , define

$$v(\alpha_U, P) := E_{\text{cell}}(P) \exp\left(-\frac{2\pi}{\beta_{\text{EW}}\alpha_U}\right).$$

Run the one-loop D10 couplings from

$$\alpha_1(M_U) = \alpha_2(M_U) = \alpha_3(M_U) = \alpha_U$$

down to a scale  $\mu$  using

$$\alpha_i^{-1}(\mu; \alpha_U, P) = \alpha_U^{-1} + \frac{b_i}{2\pi} \log \frac{M_U(P)}{\mu},$$

with the printed D10 one-loop coefficients  $b_i$ . Let  $\mu_* = \mu_*(\alpha_U, P)$  be the fixed point determined by the tree-level  $Z$ -mass relation

$$\mu_* = \frac{v(\alpha_U, P)}{2} \sqrt{g_2(\mu_*)^2 + g_Y(\mu_*)^2}, \quad \alpha_Y = \frac{3}{5} \alpha_1.$$

At that fixed point define

$$t_2(\alpha_U, P) = 4\pi^2 \alpha_2(\mu_*; \alpha_U, P), \quad t_3(\alpha_U, P) = 4\pi^2 \alpha_3(\mu_*; \alpha_U, P).$$

Let  $\bar{\ell}_{\text{SU}(2)}(t)$  and  $\bar{\ell}_{\text{SU}(3)}(t)$  denote the nonabelian edge-entropy functions appearing in the D10 pixel constraint. The pixel-closure functional is

$$\mathcal{F}(\alpha_U; P) := \bar{\ell}_{\text{SU}(2)}(t_2(\alpha_U, P)) + \bar{\ell}_{\text{SU}(3)}(t_3(\alpha_U, P)) - \frac{P}{4}.$$

The printed D10 package takes  $\alpha_U(P)$  to be the branch value selected by solving

$$\mathcal{F}(\alpha_U; P) = 0.$$

Only after that forward solve are the internal transmutation parameters fixed:

$$t_U(P) := 4\pi^2 \alpha_U(P), \quad t_{\text{tr}}(P) := \frac{2\pi}{(N_c + 1)\alpha_U(P)},$$

and with them the downstream scale data

$$t_2(P) := t_2(\alpha_U(P), P), \quad t_3(P) := t_3(\alpha_U(P), P), \quad v(P) := v(\alpha_U(P), P).$$

Thus the branch runs from OPH input  $P$  to the internal transmutation data  $(t_U(P), t_{\text{tr}}(P))$ , then to the downstream local scale data  $(t_2(P), t_3(P), v(P))$ , and only then to the low-energy couplings. It does not use measured  $\alpha_i(m_Z)$  to infer those internal  $t$ -parameters.

Every quoted D10 electroweak number is downstream of this forward map. In particular,  $\alpha_{\text{em}}^{-1}(m_Z)$ ,  $\sin^2 \theta_W(m_Z)$ ,  $m_W$ , and the stage-3  $m_Z$  readout are forward-emitted outputs of the chosen D10 running/matching package. When compared with observation, they serve as validation of that printed implementation in the calibration-sector sense recorded in the premise ledger. What remains outside theorem level is a general existence/uniqueness or monotonicity proof for the scalar root selected by the printed D10 package under arbitrary convention changes.

## 8 Relation to Holography and Existing UV Frameworks

OPH belongs to the holographic family of ideas, but its primitive data are different from those used in asymptotic-boundary constructions. The distinction matters because several of the central claims of this paper, including the treatment of  $\Lambda$ , the handling of factorization, and the emergence of gauge structure, depend on it.

## 8.1 Static-patch screen versus asymptotic boundary

In asymptotic-boundary holography one starts from a dual theory at infinity and reconstructs the bulk inward. Here one starts from a finite-capacity screen equipped with a net of local patch algebras and reconstructs the effective bulk from overlap consistency. The primitive object is therefore a family of observer patches with nontrivial overlaps, not a single global boundary theory.

This shift has two technical consequences. First, subsystem factorization is treated from the beginning as a gluing problem with centers and sector labels. Second, the physical role of the horizon is local and operational: every observer has direct access only to a patch algebra, and global law is whatever survives reconciliation on overlaps.

## 8.2 Positive $\Lambda$ and finite capacity

The null-modular reconstruction of Section 6 determines the stress tensor only up to a metric term. This means that local null data do not fix  $\Lambda$ . The cosmological constant enters only when one supplies the global screen capacity

$$N_{\text{scr}} = \log \dim \mathcal{H}_{\text{tot}}.$$

The resulting relation

$$\Lambda = \frac{3\pi}{GN_{\text{scr}}}$$

fits naturally with a de Sitter static patch of finite entropy via  $N_{\text{scr}} = A_{\text{dS}}/(4G)$ . The local null-modular reconstruction leaves the metric ambiguity, while screen capacity supplies the global datum that fixes  $\Lambda$ . OPH is therefore formulated in a de Sitter-first language. The positive cosmological constant is tied to finite capacity rather than to a deformation of a negative- $\Lambda$  starting point.

## 8.3 Factorization, gauge structure, and the string sector

The factorization problem of gauge theory and gravity is often treated as a technical nuisance. Here it is part of the architecture. Edge-center completion turns the collar center into a first-class object, and the entropy split

$$S_{\text{gen}} = \langle LC \rangle + S_{\text{bulk}}$$

is then a structural consequence rather than an added prescription.

In the bosonic EFT branch, compact gauge reconstruction follows only after the explicit transportable sector-colimit and the bosonic ordinary/central gauge premises of Assumption 3.7 are supplied. The genuinely noncentral fixed-cutoff branch is handled separately by crossed-module data rather than by the ordinary compact-group theorem. The Standard Model quotient is then selected by minimal admissibility. This differs sharply from approaches in which the gauge group is specified in advance.

The string-theory relation is similarly reversed. The edge partition function

$$Z_{\text{edge}}(t) = \sum_R d_R^2 e^{-tC_2(R)}$$

is the closed partition function obtained after gluing the open-edge weights  $p_R(t) \propto d_R e^{-tC_2(R)}$ . Peter–Weyl identifies this sum with the compact-group heat kernel at the identity,  $Z_{\text{edge}}(t) = K_t(1)$ , and the Chapman–Kolmogorov law gives the corresponding collar-sewing rule. If a large- $N_{\text{edge}}$  regime exists, with  $N_{\text{edge}}$  distinct from the physical color rank  $N_c = 3$ , its free energy admits a genus expansion. Here this is read as a conditional effective reorganization of edge degrees of freedom, not as a derived critical-superstring completion. Any further lift to critical superstring structure

would additionally require worldsheet supersymmetry, critical dimension, modular invariance, and GSO projection beyond the OPH axiom chain.

Feature	Asymptotic-boundary holography	OPH
Primitive data	Boundary theory at infinity	Finite-capacity screen with observer-patch net
Factorization across cuts	Subtle and often indirect	Explicit edge-center completion with central labels
Status of $\Lambda$	External background datum	Fixed globally by $N_{\text{scr}}$ after local null reconstruction
Gauge group	Usually specified as part of the model	Reconstructed from edge sectors, then filtered by admissibility
String sector	Fundamental dual description	Effective large- $N_{\text{edge}}$ reorganization of edge dynamics

## 9 Claim Tiers and Falsifiability

The recovered core is

$$(D1-D5) \cup (D7-D9),$$

namely the relativity chain together with the realized Standard Model structural chain. D6 is a separate input-dependent capacity corollary, D10 is the integrated calibration sector, and D12 collects phenomenological continuations.

### 9.1 Prediction audit by claim tier

Tier	Representative outputs	What would actually falsify the tier
Phase I recovered core (Theorem 3.9; D1–D5, D7–D9)	Confluence of overlap repair, conditional Lorentz kinematics, the conditional Jacobson-type Einstein branch in the stated scaling regime, compact gauge reconstruction in the bosonic branch, the realized Standard Model quotient chain, the exact hypercharge lattice on the realized matter package, the realized counting chain $N_g = 3$ then $N_c = 3$ , and the product-group consequence of no gauge-mediated proton decay	A mathematical failure in the derivation chain, or data that require a different realized gauge quotient, different realized hypercharges, a different color or generation count on the admissible branch, or gauge-mediated proton decay
Phase II input-dependent corollary (D6)	$\Lambda = \frac{3\pi}{GN_{\text{scr}}}$ once the screen-capacity identification is made, while any retained capacity-level neutrino side estimate on that input is legacy bookkeeping rather than the live neutrino theorem lane	Failure of the screen-capacity identification or of the specific global capacity relation. Such a failure would not by itself erase the recovered core
Phase II calibration sector (D10)	forward transmutation data $\alpha_U(P)$ , $t_U(P)$ , $t_{\text{tr}}(P)$ , pixel-closure gauge-coupling consistency, $\alpha_i(m_Z)$ , $\alpha_{\text{em}}^{-1}(m_Z)$ , $\sin^2 \theta_W(m_Z)$ , and the target-free source-only electroweak quintet $(W, Z, \alpha_{\text{em}}^{-1}, \sin^2 \theta_W, v)$ on the D10 calibration branch	Failure of the integrated calibration closure once $P$ and the printed running/matching conventions are imposed

Tier	Representative outputs	What would actually falsify the tier
Phase III phenomenological continuations (D12 and beyond)	Flavor ansätze, charged-lepton continuation ansätze beyond exact centered readback, texture branches, the weighted-cycle / Majorana-holonomy neutrino branch above the legacy D6 side estimate, further neutrino mass/mixing refinements, dark-sector response laws, baryogenesis estimates, proton-spin bookkeeping, proton-lifetime estimates beyond the gauge-channel exclusion, black-hole spectroscopy templates, conditional string/worldsheet reorganizations, conjectural critical-superstring extensions, and other downstream phenomenology	Retraction of the corresponding continuation only. These branches are not part of the recovered core and their failure is not, by itself, a falsification of relativity-plus-Standard-Model recovery

Phase I is the recovered-core tier, Phase II is adjacent but non-core, and Phase III contains phenomenological continuations. Interpretive strange-loop discussions lie outside these tiers: the paper does not define an OPH closure map on an invariant observer-supporting sector or prove uniqueness or stability for such a map.

## 9.2 Recovered-core discriminators

Once the claim boundary is made explicit, the sharpest discriminators are the ones that do not depend on auxiliary flavor, dark-sector, baryogenesis, or string ansätze.

### 9.2.1 Realized Standard Model gauge structure

On the realized MAR-admissible branch, this paper fixes

$$G_{\text{phys}} = \frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}, \quad N_g = 3, \quad N_c = 3,$$

with the exact one-generation hypercharge lattice. Here the realized counting chain is sequential: the CP/UV admissibility window and minimal admissibility yield  $N_g = 3$ , and Witten's SU(2) anomaly parity with that realized generation count yields  $N_c = 3$ .

$$Y_Q = \frac{1}{6}, \quad Y_L = -\frac{1}{2}, \quad Y_u = -\frac{2}{3}, \quad Y_d = \frac{1}{3}, \quad Y_e = 1, \quad Y_H = \frac{1}{2}.$$

Evidence that the realized low-energy gauge structure or realized hypercharges differ from these values would directly contradict the recovered core.

### 9.2.2 Product-group corollary

Because the realized gauge structure is a product group up to the finite central quotient fixed above, there are no mixed  $(3, 2, \pm 5/6)$  gauge bosons. Hence

$$\tau_p^{(\text{gauge})} = \infty$$

for gauge-mediated proton decay. Observation of proton decay specifically attributable to gauge-boson exchange would therefore falsify the realized product-group branch.

### 9.2.3 Relativity branch

The relativity claim is conditional but still sharp about its scope: if the stated geometric-modular, null-bridge, and fixed-cap-stationarity premises are realized in the intended scaling regime, then this paper predicts Lorentz kinematics on the screen and a Jacobson-type Einstein branch. What is being tested here is the existence of that scaling regime and its premises, not a fixed-cutoff matrix-algebra identity.

### 9.3 What does *not* count as a contradiction of the recovered core

The following do *not* falsify Theorem 3.9 if they fail in their present form:

1. the uniform  $\mathbb{Z}_6$  center-label ensemble and the associated  $\varepsilon = 1/6$  flavor ansatz;
2. charged-lepton continuation ansätze beyond the exact centered-readback / common-shift frontier, texture exponents, or legacy capacity-level neutrino side estimates;
3. modular-anomaly dark-sector response ansätze, including MOND/RAR-style scaling claims;
4. baryogenesis estimates based on extra suppression-counting assumptions; a baryogenesis theorem would still need a concrete out-of-equilibrium mechanism, derived CP-odd source terms, justified defect/sphaleron counting, washout control, and a freeze-out computation of the final asymmetry;
5. proton-spin ansätze or proton-lifetime estimates beyond the structural exclusion of gauge-mediated decay;
6. discrete-horizon spectroscopy templates;
7. large- $N_{\text{edge}}$  string/worldsheet reorganizations.

These are all post-Phase-I, non-core branches. Some are Phase-II input-dependent estimates, while the rest are Phase-III continuations. None is part of this paper's recovered relativity-plus-Standard-Model core.

## 10 Common Objections and Clarifications

### 10.1 Is the use of $P$ circular?

The pixel area  $P$  is the external scale input for the separate gauge-coupling sector used here, and the circular step that read the internal D10 transmutation parameter back from measured low-energy couplings is omitted from this paper's presentation. Within the printed D10 implementation,  $P$  first fixes  $M_U(P)$  and  $E_{\text{cell}}(P)$ , then a one-dimensional pixel-closure solve fixes  $\alpha_U(P)$ , equivalently the internal transmutation data  $t_U(P)$  and  $t_{\text{tr}}(P)$ , and only then do  $t_2(P)$ ,  $t_3(P)$ ,  $v(P)$ , and the running electroweak outputs appear. Quantities algebraically entangled with that calibration remain on the Phase-II implementation surface: they are forward-emitted there, and comparison with observation checks that printed implementation rather than enlarging the recovered-core claim set.

Operationally, changing  $P$  moves the calibration family  $(\alpha_U, \alpha_i(m_Z), \sin^2 \theta_W(m_Z))$  through that forward solve. It does not alter the parameter-free structural outputs such as the gauge quotient or hypercharge lattice. Downstream matter-sector continuations, including charged-lepton continuation ansätze beyond the exact centered-readback / common-shift frontier, remain outside this SM/GR derivation paper's recovered-core theorem package.

## 10.2 Does a fixed UV cell size break Lorentz invariance?

The UV regulator is placed on the screen algebra, not on an emergent bulk spacetime lattice. Lorentz kinematics arise from the continuum modular action on caps, summarized in Corollary 6.5, rather than from microscopic invariance of a bulk discretization.

This is conceptually similar to other continuum limits in statistical mechanics and lattice field theory. A regulator may break a symmetry at finite cutoff while the infrared fixed point restores it exactly. Here the restoration is stronger than a generic RG expectation because the modular-flow theorem identifies the symmetry group itself:

$$\text{Conf}^+(S^2) \cong \text{SO}^+(3,1).$$

Residual Lorentz violation would therefore have to arise from a failure of the regulator premises or from a failure of the modular-flow theorem, not merely from the existence of a UV cell size. The remaining UV-side burden on the Lorentz branch is therefore the explicit scaling-limit plus geometric-subnet extraction-and-rigidity package under **T2**, not the mere existence of a finite cell size.

## 10.3 Why use type-I algebras if continuum QFT uses type III?

Type I is a regulator statement. At every finite cutoff the patch and collar algebras are finite-dimensional matrix algebras, so entropy, recovery maps, and

$$K = -\log \rho$$

are literal finite-dimensional objects. The Lorentz branch is *not* a claim that the continuum observer algebra stays in that class. The refinement picture is

$$\text{finite type-I regulator net} \longrightarrow \text{scaling-limit observer net } \mathcal{A}_\infty,$$

and the limit can leave the regulator class. Axiom 3.3 controls the realized state-side branch across refinement; it does *not* by itself prove that the refinement limit remains type I or that the emitted scaling-limit pair is a vacuum or canonical cap state.

On the geometric-subnet branch of Theorem 6.3, the scaling-limit cap algebras may be non-type-I and, in the continuum-QFT case of interest, are expected to be type III. Then there is generally no cap density matrix  $\rho_C \in \mathcal{A}_\infty^{\text{geo}}(C)$  and no bounded operator  $K_C = -\log \rho_C$  inside the limit algebra. The correct continuum object is the modular automorphism group of the pair  $(\mathcal{A}_\infty^{\text{geo}}(C), \omega_\infty^{\text{geo},C})$ , which on that branch acts geometrically and typically outerly.

There is therefore no contradiction between using type-I regulators and aiming at a type-III continuum. The finite regulator provides the collar decomposition, recovery control, and carried errors. What the fixed-cutoff MaxEnt package adds is control of the realized state-side branch through one common finite-dimensional multiplier family under refinement. What it still does *not* prove is the algebraic type of the continuum limit or that the limit state is a vacuum or canonical BW cap state. The phrase “BW/canonical cap phase,” when used in this paper, means only the case in which the realized refinement branch emits a scaling-limit geometric cap pair with the standard geometric modular automorphism group. In the generic non-type-I case that is an outer geometric modular action, not an inner density-matrix identity.

The clean internal extension route remains a two-step chain, and the first object is the pressure point. First, one needs a canonical scaling-limit observer cap-pair realization on the geometric subnet from transported marginals. The theorem contract for that object is explicit: the realized transported cap-local system packages the cap-local test family on  $\mathcal{A}_n^{\text{geo}}(C_n)$ , the projectively

compatible transported marginal family, and the asymptotic transport-equivalence certificate. The remaining emitted witness is a derived vanishing carried-collar schedule on fixed local collars, explicitly decomposed into the constructive-recovery remainder and the faithful modular-defect term, namely the convergence to zero of the carried quantity

$$r_{\text{FR}}(\varepsilon_{n,m,\delta}) + 4\lambda_*^{-1} \delta_{A_{n,m,\delta}:B_{n,m,\delta}:D_{n,m,\delta}}^{\text{M}}(\varepsilon_{n,m,\delta})$$

after transport to each fixed local collar model. Only after that can local weak-\* extraction and GNS gluing emit the realized scaling-limit geometric cap pair.

Second, one needs ordered cut-pair rigidity on that realized limit. That stage would collapse the remaining cap-preserving conformal freedom to the unique BW hyperbolic subgroup, after which the modular KMS condition fixes the  $2\pi$  normalization. Ordered cut-pair rigidity remains symbolic until the realized scaling-limit geometric cap pair exists. Until those two objects are internalized, every stronger BW / vacuum / canonical identification remains explicitly theorem-external shorthand for the geometric-subnet branch of Theorem 6.3, not a derived OPH theorem.

This is analogous to ordinary lattice field theory: one computes at finite regulator in a type-I algebra and then asks which continuum algebraic phase the scaling limit realizes. The additional structural point here is that the realized state-side branch is controlled under refinement. The open problem is to identify which scaling-limit observer algebra/state pair the realized branch produces and then to prove BW rigidity on that pair.

## 10.4 UV branch and scaling-limit scope

At fixed cutoff, Axiom 3.3 yields a quasi-local finite-range interacting branch, and Propositions 4.8 and 4.10 show that the physical normal form is unique only modulo quotient-level OPH-stable equivalence, while the terminal expectation functionals on the declared fixed-point / quotient-local physical algebras are already schedule-independent on that carrier even when microscopic representatives differ by gauge labels globally or by sector labels on one regional quotient-local glued state. The invariant fixed by the axiom language is the class  $[\mathcal{U}]_{\text{OPH}}$  of the physical branch modulo implementation hiding and inert ancillary stabilization, not a unique microscopic representative. The genuinely noncentral topological case is also closed at fixed cutoff by the higher-gauge package of Theorems 5.3–5.6 and Proposition 5.7, so the fixed-cutoff topological UV surface is closed on the ordinary, central-defect, and crossed-module branches alike. Beyond fixed cutoff, the continuum modular/geometric lift remains the geometric-subnet extraction-plus-rigidity problem under **T2**, and the transportable bosonic gauge lift still carries Assumption 3.7 when invoked. The consensus paper also classifies that D1 lane explicitly: on each fixed finite patch net, normal-form computation is finite-state and decidable with the Lyapunov step bound, the current approximate-stability results are only the collar-local splice and record controls, and no universality theorem for a uniform family of growing patch nets and repair laws is part of the emitted theorem surface.

## 10.5 Why is charge quantization possible without a simple GUT?

The framework uses the global quotient and anomaly structure rather than a simple-group embedding. Once the realized matter content is fixed, the subgroup acting trivially on all states is exactly  $\mathbb{Z}_6$ , and the anomaly equations force the sixth-integer hypercharge lattice. Integer electric charge for color singlets is then a property of the realized quotient structure. A simple unified gauge group is not required.

## 11 Discussion

The framework is strongest where the outputs are discrete or structurally rigid:

1. the Lorentz branch on the extracted prime geometric cap pair once the theorem-local BW lift objects are supplied;
2. the conditional scaling-limit Einstein branch under fixed-cap stationarity and the null modular bridge together with the bounded-interval projective branch;
3. the Standard Model gauge quotient chain;
4. exact hypercharges on the realized one-generation matter package with one Higgs doublet;
5. the realized counting chain  $N_g = 3$  then  $N_c = 3$  on the realized MAR-admissible branch;
6. the compact two-input quantitative layer, kept explicit as a secondary layer rather than as part of Phase I.

The local MaxEnt branch gives a finite-range interacting fixed-cutoff dynamics, Propositions 4.8 and 4.10 fix uniqueness of the UV branch only modulo OPH-stable equivalence, and Theorems 5.3–5.6 with Proposition 5.7 close the genuinely noncentral topological branch at fixed cutoff. The open items on the UV side are the continuum BW/geometric lift on the extracted prime geometric subnet under **T2** and the fuller transportable compact-gauge lift under Assumption 3.7. The smallest internal extension route to the BW/geometric lift is the two-step chain of a scaling-limit cap-pair extraction theorem that extracts  $(\mathcal{A}_\infty^{\text{geo}}(C), \omega_\infty^{\text{geo}, C})$  as local weak-\* limits of transported cap marginals on  $\mathcal{A}_n^{\text{geo}}(C_n)$  with vanishing carried collar errors, followed by an ordered null cut-pair rigidity theorem that collapses the residual cap-preserving conformal freedom on those extracted cap pairs to the unique BW hyperbolic subgroup. Beneath the derived carried-collar witness, the sharper raw datum is a transported fixed-local-collar Markov/faithfulness package: collarwise conditional mutual information vanishing together with an eventual common floor on the finite modular-transport family on each fixed local collar model. Within that lower datum, the single missing clause is the modular-transport common floor, which feeds the faithful modular-defect term. No separate exact-Markov-reference faithfulness datum is missing: on one fixed collar model, the comparison marginals inherit the same eventual floor once the exact-Markov modulus goes to zero. The corpus does not derive a refinement-uniform common floor on the finite modular-transport marginals; this single clause remains external to the emitted theorem chain. That two-step chain and its transported fixed-local-collar datum are not part of the proved theorem surface stated here.

**Shared excitation dictionary and flavor continuation status.** The D10 calibration sector gives integrated gauge-coupling closure on the displayed carrier. That carrier closes its own exact  $W/Z$  chart, and the promoted target-free source-only repair theorem emits the public electroweak quintet on the Phase II calibration branch; the freeze-once coherent repair surface serves only as compare-only validation. On that frozen authoritative repair surface the canonical  $W/Z$  reference pair is hit exactly, but it remains a compare-only sidecar below the target-free theorem. The separate D6 cosmological-capacity relation remains an input-dependent corollary. On the flavor side the manuscript treats one shared OPH excitation dictionary as the only proof-facing family datum; lane-specific integers are not introduced independently. For each realized same-label refinement arrow  $e$  on the three-generation bundle, let  $\omega_e$  be the same-label overlap holonomy scalar, let

$$d_e := 1 - \omega_e$$

be the derived edge defect, let  $g_e$  be the same-label gap scalar, and define

$$q_e := \sqrt{g_e d_e}, \quad \eta_e := \log q_e - \frac{1}{3} \sum_f \log q_f, \quad \mu_e := \frac{e^{\eta_e}}{\frac{1}{3} \sum_f e^{\eta_f}}.$$

These are the common descendants of overlap holonomy and edge-defect data. They exist once the same-label gap and overlap witnesses are fixed, and they are the shared family-side input used by the charged, quark, and neutrino continuations.

To include the charge-side bookkeeping explicitly, let  $s_e$  denote the realized sector-charge step carried by the same-label arrow  $e$ . For any lane-specific readout  $\gamma$  on the realized same-label transport graph, let

$$m_{\gamma,e} \in \mathbb{Z}_{\geq 0}$$

be the multiplicity with which  $\gamma$  traverses the elementary same-label excitation arrow  $e$ . Equivalently, along the realized transport path one has

$$s_\gamma = \sum_e m_{\gamma,e} s_e.$$

The associated excitation weight and excitation action are

$$A_\gamma := - \sum_e m_{\gamma,e} \log q_e = - \log \left( \prod_e q_e^{m_{\gamma,e}} \right), \quad Y_\gamma^{\text{exc}} := \prod_e q_e^{m_{\gamma,e}}.$$

So overlap holonomy, edge defects, gap data, and sector-charge transport determine the suppression law before any base choice is made. What earlier D12 language called a “texture exponent” or “defect count” is therefore not an extra primitive beyond the structural branch. The primitive flavor-side integers are the transport multiplicities  $m_{\gamma,e}$ . A one-number exponent in “units of  $\log 6$ ” is only the compressed summary

$$n_\gamma^{(6)} := A_\gamma / \log 6,$$

and it becomes a literal defect count only after the additional uniform sixfold center-label collapse compatible with the realized

$$\frac{\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)}{\mathbb{Z}_6}$$

quotient. Without that extra collapse, the derived OPH data are the multiplicity vector  $m_{\gamma,\bullet}$  and the excitation action  $A_\gamma$ , not an independently postulated flavor integer.

The imported realized-branch premise here is the D9 counting chain  $N_g = 3$  then  $N_c = 3$ . This shared dictionary feeds the downstream flavor lanes. The intrinsic neutrino lane factors through the same-label scalar certificate  $(g_e, \omega_e)$  and its centered descendants  $(q_e, \eta_e, \mu_e)$ ; the weighted-cycle branch therefore uses no extra primitive flavor labels. The quark mass-side continuation compresses the same dictionary to

$$(\Delta_{ud}^{\text{overlap}}, \eta_Q^{\text{centered}})$$

and, on the compact same-family branch, to the emitted ray  $\mathcal{R}_{D12}^{ud}$ . On the CKM/CP side the same-label left-handed orbit collapses to the singleton  $\sigma_{\text{ref}}$ , so the selected D12 sheet is fixed and is the wrong CKM sheet. The quark frontier is therefore layered. On the broader D12 continuation surface, the smaller honest theorem-side primitive is the light-quark isospin-breaking / overlap-defect scalar  $\Delta_{ud}^{\text{overlap}}$ : once that scalar is emitted, odd-budget neutrality fixes the light-sector pure- $B$  payload pair and the  $\tau$ -pair algebraically on the ordered three-point family. On the emitted same-family ray, that same scalar is equivalently the downstream one-scalar law `quark_d12_t1_v`

blue\_law, with intrinsic\_scale\_law\_D12 retained only as the derived wrapper. The charged sector is equally tied to the same backbone: once a charged source pair  $(\eta_{\text{ch}}, \sigma_{\text{ch}})$  exists on the ordered charged carrier  $(-1, x_2, 1)$ , the centered charged logs are fixed algebraically by

$$\begin{aligned} e_{\log, \text{centered}} &= -\frac{(3 + x_2)\sigma_{\text{ch}} - \eta_{\text{ch}}}{6}, \\ \mu_{\log, \text{centered}} &= \frac{x_2\sigma_{\text{ch}} - \eta_{\text{ch}}}{3}, \\ \tau_{\log, \text{centered}} &= \frac{(3 - x_2)\sigma_{\text{ch}} + \eta_{\text{ch}}}{6}. \end{aligned}$$

so no separate charged-family integers remain to be chosen there.

The remaining burdens sit strictly above the shared dictionary. For charged leptons, the centered common-shift quotient is already closed negatively, and the remaining theorem-grade waiting set is the promotion

$$\widehat{C}_e^{\text{cand}} \longrightarrow \widehat{C}_e$$

and then the post-promotion lift whose descended scalar is  $\mu_{\text{phys}}(Y_e)$ , with the physical identity-mode equalizer as the exact smaller forcing object beneath that scalar. For quarks, the broader honest continuation frontier is the light-quark isospin-breaking / overlap-defect scalar  $\Delta_{ud}^{\text{overlap}}$ ; on the emitted D12 mass ray this is equivalently the downstream one-scalar law `quark_d12_t1_value_law`, with `intrinsic_scale_law_D12` as the derived wrapper. For neutrinos, the same-label scalar certificate closes beneath the weighted-cycle branch, the bridge-rigidity theorem emits

$$C_\nu = G^2 Q S^{-1/2} = 0.9994295999075177,$$

with emitted proxy

$$P_\nu = 6.699825740519345,$$

while the induced paper-facing amplitude parameterization is

$$B_\nu = P_\nu C_\nu = 6.696004159297337.$$

Thus the shared excitation dictionary is not the missing theorem object. It is the common OPH family base, while old base-6 exponents survive only as compare-only compressions of its excitation action.

**Exact non-hadron output lane.** For reader-facing exact hits, the non-hadron surface is explicit and lane-split. The structural carrier chain

$$\text{Axioms 1-5} \longrightarrow D7-D9 \longrightarrow (m_\gamma, m_g, m_{\text{grav}}) = (0, 0, 0)$$

is theorem-grade structural exactness. The electroweak calibration chain

$$\text{Axioms 1-5} \longrightarrow D7-D10$$

has an exact frozen authoritative repair sidecar

$$(M_W, M_Z) = (80.377, 91.18797809193725) \text{ GeV}$$

on the frozen authoritative repair surface, but that exact pair remains compare-only beneath the target-free source-only theorem. The D11 Higgs/top branch carries a closed one-scalar forward seed with

$$m_H = 125.218922 \text{ GeV}, \quad m_t = 172.388646 \text{ GeV}.$$

The same D11 Jacobian also has a compare-only exact inverse slice at the canonical Higgs/top reference pair

$$(m_H, m_t) = (125.1995304097179, 172.3523553288311) \text{ GeV.}$$

The charged lane has an exact same-family witness

$$(m_e, m_\mu, m_\tau) = (0.00051099895, 0.1056583755, 1.7769324651340912) \text{ GeV}$$

on the same ordered eigenvalue family, but it remains same-family-only beneath the exact centered-readback / closed-common-shift frontier and the open theorem chain  $\widehat{C}_e^{\text{cand}} \rightarrow \widehat{C}_e$  together with the post-promotion lift whose descended scalar is  $\mu_{\text{phys}}(Y_e)$ . The quark lane has an exact same-family witness

$$(u, d, s, c, b, t) = (0.00216, 0.00470, 0.0935, 1.273, 4.183, 172.3523553288311) \text{ GeV,}$$

again same-family-only, because the selected D12 sheet is the wrong CKM branch, the broader honest frontier is the light-quark overlap-defect scalar  $\Delta_{ud}^{\text{overlap}}$ , and on the emitted ray this is equivalently the downstream one-scalar law `quark_d12_t1_value_law`. The neutrino lane has the weighted-cycle bridge-rigid absolute family

$$(m_1, m_2, m_3) = (0.017454720257976796, 0.019481987935919015, 0.05307522145074924) \text{ eV,}$$

$$\Delta m_{21}^2 = 7.488059465106851 \times 10^{-5} \text{ eV}^2,$$

$$\Delta m_{31}^2 = 2.5123118727618473 \times 10^{-3} \text{ eV}^2,$$

$$\Delta m_{32}^2 = 2.4374312781107786 \times 10^{-3} \text{ eV}^2.$$

The exact two-parameter positive-segment adapter remains on disk only as a diagnostic sidecar beneath the bridge-rigidity invariant  $C_\nu$  and its induced paper-facing amplitude  $B_\nu$ . The exact non-hadron output lane is complete, on a mixed theorem/compare-only/same-family surface rather than one uniform theorem tier.

**Charged-lepton continuation status.** The charged-lepton lane is not a theorem-level mass closure, and Koide-style fitting language does not describe its theorem boundary. What is exact is narrower. On the ordered charged carrier

$$(-1, x_2, 1), \quad x_2 = -0.5175863354681689,$$

every charged source pair  $(\eta, \sigma)$  determines the centered charged logs by

$$e_{\log, \text{centered}} = -\frac{(3+x_2)\sigma - \eta}{6}, \quad \mu_{\log, \text{centered}} = \frac{x_2\sigma - \eta}{3}, \quad \tau_{\log, \text{centered}} = \frac{(3-x_2)\sigma + \eta}{6}.$$

So the centered charged hierarchy closes once a charged source pair is emitted.

The absolute frontier is explicitly two-layered. First, the promotion

$$\widehat{C}_e^{\text{cand}} \longrightarrow \widehat{C}_e$$

waits on the upstream package `oph_generation_bundle_branch_generator_splitting`. Its sharp remaining clause is the descended-commutator statement `compression_descendant_commutator_vanishes_or_is_uniformly_quadratic_small_after_central_split`. The corpus proves neither exact vanishing nor uniform quadratic smallness of that descended commutator, so  $\widehat{C}_e^{\text{cand}}$  is not promoted.

Second, the common-shift quotient surface is closed negatively: the equalizer/common-shift route does not determine an absolute charged scale from centered data alone. The post-promotion slot is the refinement-stable uncentered trace lift. Its descended scalar is  $\mu_{\text{phys}}(Y_e)$ , and the exact smaller forcing object beneath that scalar is `charged_physical_identity_mode_equalizer`, namely the physical identity-mode equalizer  $\delta(r, r') = 0$  on common physical fibers. If a theorem-grade  $\widehat{C}_e$  and refinement-stable physical  $Y_e$  are obtained, then the uncentered lift, determinant-line section, and affine anchor follow canonically from that one scalar:

$$\widetilde{C}_e(Y_e) = \widehat{C}_e(Y_e) + \mu_{\text{phys}}(Y_e) \mathbf{1}, \quad s_{\text{det}}(Y_e) = 3\mu_{\text{phys}}(Y_e), \quad A_{\text{ch}}(Y_e) = \mu_{\text{phys}}(Y_e).$$

Hence the public claim boundary is exact same-carrier readback plus the closed common-shift no-go, with the remaining burdens carried first by the splitting package `oph_generation_bundle_branch_generator_splitting` and then by the post-promotion refinement-stable uncentered trace lift, with `charged_physical_identity_mode_equalizer` beneath  $\mu_{\text{phys}}(Y_e)$ . No current-corpus theorem emits public charged masses,  $g_e$ , or  $\Delta_e^{\text{abs}}$ . If a future universal charged-mass law from the external calibration input  $P$  exists, it cannot enter through the present centered same-carrier readback surface, because that surface is common-shift invariant; it would have to descend through the still-missing affine object  $A_{\text{ch}}(Y_e) = \mu_{\text{phys}}(Y_e)$ , equivalently through a theorem-grade bridge from the D10 calibration family to the charged determinant line. A target-anchored same-family exact witness reproduces the charged reference triple exactly on the same ordered eigenvalue family, and within that same-family scope the ordered three-point quadratic readout is closed through `oph_lepton_current_family_quadratic_readout_theorem`. On that same fixed target-anchored witness, the scoped affine coordinate also closes through `oph_lepton_current_family_affine_anchor_theorem`, with

$$A_{\text{ch}}^{(\text{sf})} = \frac{1}{3} \log \det(Y_e^{(\text{sf})}), \quad g_e^{(\text{sf})} = e^{A_{\text{ch}}^{(\text{sf})}};$$

this remains same-family-only and does not promote the charged theorem lane. Any balanced-circulant /  $Q = 2/3$  /  $\delta = 2/9$  relation remains compare-only continuation and is not an emitted mass theorem. This second layer is a descent contract rather than a second fitted invariant: if theorem-grade  $\widehat{C}_e$  and an admissible refinement-stable uncentered lift exist, the physical identity-mode equalizer forces the descended scalar  $\mu_{\text{phys}}(Y_e)$ .

**Quark continuation status.** The light-quark lane is a D12 continuation branch and not a recovered-core theorem. The emitted same-label left-handed solver surface closes to the singleton  $\sigma_{\text{ref}}$  (`sigma_ref`). That closure is negative:  $\sigma_{\text{ref}}$  is the wrong D12 CKM sheet, and the corpus emits no sector-attached same-label left-handed branch-changing transport from that selected sheet to the physical CKM shell. Same-sheet overlap scans, chirality-swapped basis diagnostics, and other non-sector-attached orbit improvements are compare-only and do not alter the selected-sheet statement.

The first missing theorem object is the mass-side scalar law

$$\Theta_{ud}^{\text{mass}} := \text{quark\_d12\_t1\_value\_law}.$$

On the minimal light branch

$$y_u = c_u \varepsilon^6, \quad y_d = c_d \varepsilon^6, \quad \varepsilon = \frac{1}{6},$$

so the light-quark isospin-breaking scalar is

$$\Delta_{ud}^{\text{overlap}} = \frac{1}{6} \log \frac{c_d}{c_u}.$$

The emitted same-family D12 mass object is the one-parameter ray

$$D12_{ud}^{\text{mass}} \equiv \mathcal{R}_{D12}^{ud} \quad (\text{paper id D12\_ud\_mass\_ray}),$$

and on that emitted ray the same scalar is equivalently the one-scalar law  $\Theta_{ud}^{\text{mass}}$ , identifying the ray modulus with  $t_1$ . The exact scalar identities are

$$\Delta_{ud}^{\text{overlap}} = \frac{t_1}{5}, \quad \log \frac{c_d}{c_u} = \frac{6}{5} t_1, \quad t_1 = 5 \Delta_{ud}^{\text{overlap}} = \frac{5}{6} \log \frac{c_d}{c_u}.$$

The full D12 mass-side package is then functorial:

$$\eta_Q^{\text{centered}} = -\frac{1-x_2^2}{27} t_1, \quad \kappa_Q = -\frac{t_1}{54}, \quad x_2 = -0.5175863354681689.$$

The odd source package is also forced:

$$\beta_{u,\text{diag},B}^{\text{source}} = \frac{t_1}{10},$$

$$\beta_{d,\text{diag},B}^{\text{source}} = -\frac{t_1}{10},$$

$$\text{source\_readback\_u\_log\_per\_side} = \left( -\frac{t_1}{10}, 0, +\frac{t_1}{10} \right),$$

$$\text{source\_readback\_d\_log\_per\_side} = \left( +\frac{t_1}{10}, 0, -\frac{t_1}{10} \right),$$

and with the emitted spread pair  $(\sigma_u, \sigma_d) = (5.5905, 3.3049)$ ,

$$\tau_u = \frac{\sigma_d}{10(\sigma_u + \sigma_d)} t_1, \quad \tau_d = \frac{\sigma_u}{10(\sigma_u + \sigma_d)} t_1.$$

The active builder-facing burden beneath this theorem is the pure- $B$  payload pair; `intrinsic_scale_law_D12` is the derived wrapper above  $\Theta_{ud}^{\text{mass}}$ .

The second missing theorem object is the physical-sheet lift

$$\Theta_{ud}^{\text{phys}}$$

to the same-label left-handed physical carrier

$$\Sigma_{ud}^{\text{phys}} := \left\{ (\sigma_{\text{id}}, \tau, U_{u,L}, U_{d,L}, V_{\text{CKM}}, I_{\text{CKM}}) : V_{\text{CKM}} = U_{u,L}^\dagger U_{d,L} \right\} / \sim,$$

where

$$(U_{u,L}, U_{d,L}, V) \sim (U_{u,L} D_u, U_{d,L} D_d, D_u^\dagger V D_d)$$

for diagonal  $D_u, D_d \in U(1)^3$ . Equivalently, this is a sector-attached section of the common-refinement frame-overlap quotient into  $\Sigma_{ud}^{\text{phys}}$ . The negative selector closure to  $\sigma_{\text{ref}}$  forces this theorem: same-sheet rephasing preserves CKM moduli and therefore cannot move the selected branch to the physical shell.

The third missing theorem object is the target-free absolute readout

$$\Theta_{ud}^{\text{abs}}$$

on the physical sheet. Its formula package is fixed on the current-family selected sheet. Define

$$\sigma_{\text{seed}}^{ud} := \frac{\sigma_u + \sigma_d}{2}, \quad \eta_{ud} := \frac{\sigma_u - \sigma_d}{2},$$

$$A_{ud} := \frac{1}{2(1 + \rho_{\text{ord}} - x_2^2)}, \quad B_{ud} := \frac{1}{2\left(1 - x_2^2 - \frac{x_2^2}{1 + \rho_{\text{ord}}}\right)}, \quad \rho_{\text{ord}} = 1.294284936377706.$$

Then

$$g_u = g_{\text{ch}} \exp(-(A_{ud}\sigma_{\text{seed}}^{ud} - B_{ud}\eta_{ud})), \quad g_d = g_{\text{ch}} \exp(-(A_{ud}\sigma_{\text{seed}}^{ud} + B_{ud}\eta_{ud})).$$

Once  $(g_u, g_d)$  are emitted on the physical sheet, the ordered three-point quadratic readout emits the sextet algebraically.

The exact closure contract on this quark lane is therefore three-step:

$$\begin{aligned} \Theta_{ud}^{\text{mass}} &\implies (\Delta_{ud}^{\text{overlap}}, \eta_Q^{\text{centered}}, \kappa_Q, \tau_u, \tau_d), \\ \Theta_{ud}^{\text{phys}} &\implies \sigma_{ud}^{\text{phys}} = (\sigma_{\text{id}}, \tau, U_{u,L}, U_{d,L}, V_{\text{CKM}}, I_{\text{CKM}}), \\ \Theta_{ud}^{\text{abs}} &\implies (g_u, g_d) \implies (m_u, m_d, m_s, m_c, m_b, m_t). \end{aligned}$$

**Maximal theorem-emitted quark package on the present premise set.** Let

$$\begin{aligned} P &:= p_a \\ &+ (\text{Axioms 1-5}) \\ &+ \text{Assumption 6} \\ &+ \text{the emitted OPH nodes D1-D10} \\ &+ \text{the current public D12 quark objects.} \end{aligned}$$

Then the present corpus theorem-emits exactly the following quark-side package:

$$\begin{aligned} \text{(i)} \quad D12_{ud}^{\text{mass}} = \mathcal{R}_{D12}^{ud} &= \left\{ \lambda \left( \frac{1}{5}, -\frac{1-x_2^2}{27} \right) : \lambda \geq 0 \right\}, \quad \lambda = \text{ray\_modulus} = t_1, \\ \text{(ii)} \quad \sigma_{ud} &= \sigma_{\text{ref}} \end{aligned}$$

on the emitted same-label left-handed local solver surface, with canonical token

$$D12::\text{same\_label\_left}::\text{reference\_sheet}.$$

This is a negative closure because  $\sigma_{\text{ref}}$  is the wrong D12 CKM sheet.

$$\begin{aligned} \text{(iii)} \quad g_{\text{ch}} &= 0.9231656602589082 \quad (\text{shared\_budget\_only}), \\ \sigma_u &= 5.5905, \quad \sigma_d = 3.3049, \quad \sigma_{\text{seed}}^{ud} = 4.4477, \quad \eta_{ud} = 1.1428, \\ A_{ud} &= 0.2467442927388686, \quad B_{ud} = 0.8125616987157023, \\ g_u &= 0.7797392875757557, \quad g_d = 0.12172551081512113 \quad (\text{current\_family\_only}). \end{aligned}$$

No stronger physical closure follows from this theorem-emitted package:

$$P \not\prec \Theta_{ud}^{\text{mass}}, \quad P \not\prec \Theta_{ud}^{\text{phys}}, \quad P \not\prec \Theta_{ud}^{\text{abs}}.$$

On the mass side, the emitted ray carries no theorem-grade nonzero value of its modulus  $t_1$ . On the mixing side, same-sheet rephasing preserves CKM moduli, so the emitted selector value  $\sigma_{\text{ref}}$  carries no sector-attached same-label left-handed lift to the physical CKM shell. On the absolute side, the affine sector-mean split is emitted only on the restricted current-family/shared-budget surfaces, so it carries no target-free physical-sheet pair  $(g_u, g_d)$ . Hence the exact same-family sextet and the D12 internal backread scalar package are sidecar representatives on their declared surfaces, not  $P$ -emitted physical-sheet theorem outputs.

**Minimal-extension closure theorem.** The exact minimal extension triple above that theorem-emitted package is

$$\begin{aligned} H_{\text{mass}} &: \ell_{ud} := \log \frac{c_d}{c_u}, \\ H_{\text{phys}} &: s_{ud}^{\text{phys}} : \mathfrak{M}_{ud}^{\text{CR,phys}} \rightarrow \Sigma_{ud}^{\text{phys}}, \\ H_{\text{abs}} &: A_q^{\text{phys}} : \Sigma_{ud}^{\text{phys}} \rightarrow \mathbb{R}. \end{aligned}$$

With these three objects adjoined, the physical quark lane closes formally. First,

$$t_1 = \frac{5}{6} \ell_{ud},$$

so  $H_{\text{mass}}$  emits  $\Theta_{ud}^{\text{mass}}$  and therefore fixes

$$\Delta_{ud}^{\text{overlap}} = \frac{t_1}{5}, \quad \eta_Q^{\text{centered}} = -\frac{1-x_2^2}{27} t_1, \quad \kappa_Q = -\frac{t_1}{54},$$

together with the odd package and the pure- $B$  payload pair. Second,

$$\Sigma_{ud}^{\text{phys}} := s_{ud}^{\text{phys}}([F_0^\dagger F_1])$$

emits  $\Theta_{ud}^{\text{phys}}$ . Third, writing

$$\log g_u = A_q^{\text{phys}} - (A_{ud}\sigma_{\text{seed}}^{ud} - B_{ud}\eta_{ud}), \quad \log g_d = A_q^{\text{phys}} - (A_{ud}\sigma_{\text{seed}}^{ud} + B_{ud}\eta_{ud}),$$

$H_{\text{abs}}$  emits  $(g_u, g_d)$ , hence  $\Theta_{ud}^{\text{abs}}$ , and the ordered three-point quadratic readout then emits

$$(m_u, m_d, m_s, m_c, m_b, m_t).$$

So the exact closure route is

$$P + H_{\text{mass}} + H_{\text{phys}} + H_{\text{abs}} \implies \Theta_{ud}^{\text{mass}}, \Theta_{ud}^{\text{phys}}, \Theta_{ud}^{\text{abs}} \implies (m_u, m_d, m_s, m_c, m_b, m_t).$$

A target-anchored same-family exact witness reproduces the six running quark reference masses exactly on the same ordered three-point family,

$$(u, d, s, c, b, t) = (0.00216, 0.00470, 0.0935, 1.273, 4.183, 172.3523553288311) \text{ GeV},$$

and within that same-family scope the selected-sheet exact readout closes through `oph_quark_current_family_selected_sheet_exact_closure`, built on the ordered three-point quadratic readout theorem `oph_quark_current_family_quadratic_readout_theorem`. That exact-hit sidecar does not move the selected D12 sheet and does not emit  $\Delta_{ud}^{\text{overlap}}$ , `quark_d12_t1_value_law`, or a target-free physical-sector pair  $(g_u, g_d)$ . The builder-facing pure- $B$  payload pair beneath the reported quark values is a separate open object. A separate D12 internal backread sidecar is also on disk. On that continuation-only surface, feeding the emitted reference-free forward light-quark pair through the coefficient-ratio identification  $c_u/c_d = y_u/y_d = m_u/m_d$  and the D12 normalization  $\Delta_{ud}^{\text{overlap}} = \frac{1}{6} \log(c_d/c_u)$  fixes

$$\begin{aligned} \Delta_{ud}^{\text{overlap}} &= 0.13431740757663183, & t_1 &= 0.6715870378831591, \\ \eta_Q^{\text{centered}} &= -0.018210067243314802, & \kappa_Q &= -0.012436796997836279. \end{aligned}$$

That sidecar closes the D12 mass-side scalar package numerically on its own continuation surface, but it does not replace the public theorem frontier and does not remove the wrong-sheet CKM boundary.

## 11.1 Neutrino and hadron continuation status

The isotropic intrinsic neutrino branch is ruled out by the exact atmospheric-splitting cap, so the live continuation lane is the weighted-cycle branch. On that branch the standard-math-fixed balanced transport-load selector is the arithmetic midpoint on the positive segment between  $\chi = 1 + \varepsilon$  and  $1 + \gamma/2$ :

$$D_\nu = \frac{\chi + 1 + \gamma/2}{2}, \quad p_\nu(\gamma, \varepsilon) = 1 + \gamma + \frac{\varepsilon}{D_\nu}.$$

This weighted-cycle branch repairs the oscillation-shape output to the physical PMNS/hierarchy regime on this continuation surface. On that lane the theorem-grade bridge object is the reduced invariant  $C_\nu$  above the emitted proxy  $P_\nu$ , while  $B_\nu = P_\nu C_\nu$  is retained only as the paper-facing amplitude parameterization and the older positive-segment adapters remain diagnostic-only. It emits

$$\begin{aligned} \theta_{12} &= 34.225904631810025^\circ, & \theta_{23} &= 49.72282845058266^\circ, & \theta_{13} &= 8.686355527700156^\circ, \\ \delta_{\text{PMNS}} &= 305.58061231449796^\circ, \\ J &= -0.02753115613565372, \\ \frac{\Delta m_{21}^2}{\Delta m_{32}^2} &= 0.030721110097966534. \end{aligned}$$

So the continuation branch reaches the observed oscillation window, fixes the normal-hierarchy shape, and leaves no hidden extra discrete selector on that lane.

The normalized same-label overlap-defect weight section closes beneath that weighted-cycle shape, and the paper-facing amplitude parameterization is

$$B_\nu := \lambda_\nu q_{\text{mean}}^{p_\nu} / m_{\star, \text{eV}}.$$

The emitted internal proxy is

$$P_\nu := I_\nu^{1/2} \widehat{\text{ratio}}^{1/2} \text{sum\_defect}^{-1} = 6.699825740519345.$$

Writing

$$G := \text{sum\_gap}, \quad Q := \text{prod\_qbar}, \quad S := \text{solar\_response\_over\_mstar},$$

the weighted-cycle bridge rigidity theorem identifies the physical reduced bridge invariant with the distinguished family-assisted optimizer

$$C_\nu = G^2 Q S^{-1/2} = \text{sum\_gap}^2 \text{prod\_qbar} \text{solar\_response\_over\_mstar}^{-1/2} = 0.9994295999075177.$$

Hence

$$\begin{aligned} B_\nu &= P_\nu C_\nu = 6.696004159297337, \\ \lambda_\nu &= \frac{m_{\star, \text{eV}}}{q_{\text{mean}}^{p_\nu}} P_\nu C_\nu = 1.7237014208357415. \end{aligned}$$

If  $\hat{m}_i$  and  $\widehat{\Delta m_{ij}^2}$  denote the branch-fixed scale-free eigenmasses and splittings, then

$$m_i = \lambda_\nu \hat{m}_i, \quad \Delta m_{ij}^2 = \lambda_\nu^2 \widehat{\Delta m_{ij}^2},$$

with branch-fixed coefficients

$$\hat{m} = (0.010126301485273374, 0.011302414501969324, 0.03079142408841061),$$

$$\begin{aligned}\widehat{\Delta m_{21}^2} &= 2.520259180367675 \times 10^{-5} \text{ eV}^2, \\ \widehat{\Delta m_{31}^2} &= 8.455698156217034 \times 10^{-4} \text{ eV}^2, \\ \widehat{\Delta m_{32}^2} &= 8.203672238180266 \times 10^{-4} \text{ eV}^2.\end{aligned}$$

Therefore the emitted absolute family is

$$\begin{aligned}(m_1, m_2, m_3) &= (0.017454720257976796, 0.019481987935919015, 0.05307522145074924) \text{ eV}, \\ \Delta m_{21}^2 &= 7.488059465106851 \times 10^{-5} \text{ eV}^2, \\ \Delta m_{31}^2 &= 2.5123118727618473 \times 10^{-3} \text{ eV}^2, \\ \Delta m_{32}^2 &= 2.4374312781107786 \times 10^{-3} \text{ eV}^2.\end{aligned}$$

The defect-weighted same-label edge family

$$q_e = \sqrt{g_e d_e}, \quad \eta_e = \log q_e - \frac{1}{3} \sum_f \log q_f, \quad \mu_e = \mu_\nu \frac{e^{\eta_e}}{\frac{1}{3} \sum_f e^{\eta_f}},$$

remains the best closed constructive local object beneath the bridge, while the exact two-parameter positive-segment adapter, the bridge corridor, and the reduced-correction candidate audit are retained only as diagnostic surfaces and do not feed back into theorem state. Taken together with the exact  $W/Z$  frozen repair pair, the closed Higgs/top forward seed together with its compare-only exact inverse slice, the exact charged/quark same-family witnesses, and the weighted-cycle bridge-rigid absolute neutrino family, the exact non-hadron output lane is complete, even though the theorem-grade charged and quark lanes remain open at their sharper remaining objects.

**Local unification surface and exact-release frontier.** The local quantitative bridge can be stated without inflating the recovered-core claim. On the bosonic side, the declared pixel input  $P$  fixes the D10/D11 trunk

$$P \mapsto \alpha_U(P) \mapsto (t_U(P), t_{\text{tr}}(P)) \mapsto v(P),$$

after which the D10 source basis emits the public electroweak pair and the D11 forward seed

$$\sigma_{D11, \text{HT}} = \alpha_U(P) \cos(2\theta_{W0}) / \sqrt{\pi}$$

with no inverse readback of the internal transmutation data from measured low-energy couplings. emits the Higgs/top branch. On the gravity side, the same D10 pixel law packages

$$\bar{\ell}_{\text{SU}(2)}(t_{2, \text{run}}) + \bar{\ell}_{\text{SU}(3)}(t_{3, \text{run}}) = P/4, \quad G = \frac{a_{\text{cell}}}{4 \bar{\ell}(t)}.$$

The exact local release burden consists of five explicit objects: one theorem/convention split for the familiar-unit readout, one branch-preserving D10/gravity edge-entropy bridge beneath the scalar identity

$$\bar{\ell}_{\text{shared}} = \bar{\ell}_{\text{SU}(2)}(t_{2, \text{run}}) + \bar{\ell}_{\text{SU}(3)}(t_{3, \text{run}}),$$

one strict classical-regime clause above the BW/Lorentz honesty package, one target-free D10 chart-identity theorem for exact live  $W/Z$ , and one live-forward D11 Higgs-promotion theorem. On that declared extension surface the local release values are

$$c = 299792458 \text{ m/s}, \quad G = 6.674299995910528 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2},$$

$$(M_W, M_Z, M_H) = (80.377, 91.18797809193725, 125.1995304097179) \text{ GeV.}$$

This is extension-level rather than current-corpora theorem status:  $c$  is structural from the Lorentz branch rather than fitted from  $P$ , and the displayed  $G$  row is an exact emitted branch value rather than a literal zero-difference identity against the rounded benchmark  $6.6743 \times 10^{-11}$ .

The hadron lane remains separate. It depends on one production backend export bundle with publication-complete manifest provenance, real stable-channel correlator arrays, and later nonperturbative runtime/systematics completion rather than on a closed theorem-level mass surface. The current particle roadmap does not include hadron-mass closure, because that remaining nonperturbative execution burden is not a realistic target within the current project compute scope, so the neutrino weighted-cycle improvement does not promote hadron observables.

The paper does not address simulation narratives, observer-continuation mechanisms, or complete closure of all low-energy observables.<sup>2</sup>

## 12 Conclusion

Observer-Patch Holography starts from overlap consistency of observer patches and yields, within one framework, several effective structures usually treated separately in modern fundamental physics. This SM/GR derivation paper supports a unique schedule-independent normal form for overlap repair and hence a unique physical UV branch only modulo boundary redundancy, implementation hiding, and inert ancillary stabilization, a conditional Lorentz branch on the extracted prime geometric cap pair, a conditional Jacobson-type Einstein branch under fixed-cap stationarity and the null modular bridge together with the bounded-interval projective branch, the Standard Model gauge quotient with exact hypercharges on the realized matter package under the MAR admissibility package in the bosonic internal-gauge branch, the realized counting chain  $N_g = 3$  then  $N_c = 3$  on that branch, a separate integrated gauge-calibration sector, and a separate input-dependent cosmological-capacity corollary. Phenomenological continuations are discussed separately from the recovered core.

The framework therefore presents general relativity and the Standard Model as effective descriptions arising once the stated scaling-limit, geometric-subnet, and categorical premises are realized. The local quantitative claim surface is equally explicit: on a five-object extension surface, the same declared pixel input  $P$  controls the electroweak/Higgs mass trunk and the classical-regime gravity coupling bridge, while the invariant causal speed  $c$  is the structural Lorentz output with explicit SI readout. Open items are sharper quantitative control, full derivation of the refinement-limit bosonic ordinary/central gauge package of Assumption 3.7, a fuller fermionic gauge branch, explicit microscopic realizations that emit the prime geometric cap pair and support ordered cut-pair rigidity, and proof of those five local exact-release objects on the live corpus.

## A Supplemented Gravity and Structural Gauge Details

This appendix rehouses the gravity-side and structural gauge items that had been stranded in the former technical supplement. They belong here because they sharpen the compact paper's Lorentz, Einstein, cosmological-constant, and product-group claim surfaces without widening the recovered core beyond its stated premises.

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<sup>2</sup>Questions outside this theorem package do not alter the specific claim set established here. Even a separate interpretive habitat theorem for OPH state-and-law data would not enlarge this SM/GR derivation paper's Phase-I scope unless the actual closure map, invariant admissible sector, and stability hypotheses were also proved at the same tier.

## A.1 Lorentz and Einstein Bridge

**Theorem A.1** (Conformal group isomorphism). *The orientation-preserving conformal group of  $S^2$  is isomorphic to the connected Lorentz group:*

$$\text{Conf}^+(S^2) \cong \text{PSL}(2, \mathbb{C}) \cong \text{SO}^+(3, 1).$$

**Theorem A.2** (Geometric modular flow on caps on the extracted prime geometric subnet). *Assume the OPH axioms, the derived fixed-cutoff collar package, the controlled scaling-limit scope clause **T2**, and the theorem-local realized geometric cap-pair extraction plus ordered cut-pair rigidity hypotheses on the extracted prime geometric subnet. Then the scaling-limit modular automorphism group of the cap pair is geometric conformal dilation on that subnet, with the standard  $2\pi$  normalization.*

**Lemma A.3** (Null data ambiguity). *If a symmetric tensor  $X_{ab}$  satisfies*

$$X_{ab}k^ak^b = 0$$

*for every null vector  $k$ , then  $X_{ab} = \phi g_{ab}$  for some scalar  $\phi$ .*

**Corollary A.4** (Null modular data determine Einstein only up to the metric term). *Null modular data determine  $T_{ab}$  only up to  $\phi g_{ab}$ . Consequently the Einstein equation is fixed locally only up to  $\Lambda g_{ab}$ .*

**Theorem A.5** (Conditional Jacobson-type rest-frame relation). *In the local Lorentzian scaling regime, under generalized-entropy stationarity, the half-line generator/charge identification of the null bridge, the bounded-interval transport input used in the local Lorentzian regime, the internal small-ball bridge, and admissible first variations about a maximally symmetric reference state, one obtains the rest-frame entanglement-equilibrium relation that drives the compact paper's local Einstein branch.*

**Theorem A.6** (Newton coupling from edge entropy density). *The gravitational coupling is fixed by the edge entropy density:*

$$G = \frac{a_{\text{cell}}}{4\ell(t)}.$$

## A.2 Cosmological-Constant Structural Separation

**Lemma A.7** (Vacuum energy blindness). *For any null vector  $k$ ,*

$$T_{kk}^{\text{vac}} = T_{ab}^{\text{vac}}k^ak^b = -\rho_{\text{vac}}g_{ab}k^ak^b = 0.$$

*Vacuum-energy contributions therefore lie in the kernel of the null map  $T_{ab} \mapsto T_{kk}$ .*

**Proposition A.8** (Structural separation). *Within the structural split used here:*

1. *local modular/null data fix  $T_{ab}$  only up to  $\phi g_{ab}$ ; and*
2.  *$\Lambda$  is fixed by global screen capacity, not by local null data.*

*So the large vacuum-energy bookkeeping of EFT is not itself the local quantity determining curvature on this branch.*

**Theorem A.9** (No local  $\Lambda$  prediction). *Within the null-modular reconstruction used here:*

1. *all  $T_{kk}$  data are unchanged under  $T_{ab} \rightarrow T_{ab} + \phi g_{ab}$ ;*
2. *therefore  $\Lambda$  cannot be fixed by local overlap consistency alone; and*
3.  *$\Lambda$  requires the separate global screen-capacity input.*

**Capacity and de Sitter scales.** The global-capacity closure is

$$\Lambda = \frac{3\pi}{GN_{\text{scr}}}, \quad S_{\text{dS}} = \frac{A_{\text{dS}}}{4G} = \frac{3\pi}{G\Lambda} = N_{\text{scr}}.$$

At the observed  $\Lambda \approx 1.09 \times 10^{-52} \text{ m}^{-2}$ , this gives the benchmark scales

$$r_{\text{dS}} = \sqrt{\frac{3}{\Lambda}} \approx 1.66 \times 10^{26} \text{ m}, \quad t_{\Lambda} = \frac{r_{\text{dS}}}{c} \approx 17.5 \text{ Gyr},$$

$$N_{\text{scr}} \approx 2.85 \times 10^{122}.$$

The companion pixel benchmarks are

$$a_{\text{cell}} \approx 1.63 \ell_P^2, \quad \ell_{\text{UV}} = \sqrt{a_{\text{cell}}} \approx 1.28 \ell_P, \quad \bar{\ell} \approx 0.408.$$

For comparison, the flat  $\Lambda$ CDM age formula

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_{\Lambda}}} \sinh^{-1} \left( \sqrt{\frac{\Omega_{\Lambda}}{\Omega_m}} \right)$$

gives the usual benchmark  $t_0 \approx 13.8 \text{ Gyr}$  at  $H_0 \approx 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1}$  and  $\Omega_{\Lambda} \approx 0.685$ .

### A.3 Product-Group Structural Corollaries

**Theorem A.10** (Factorization equivalence).

*Factorizing edge weights  $\iff$  Additive boundary Laplacian  $\iff$  Product gauge group.*

If

$$H_{\partial} = H_{\partial}^{(1)} + H_{\partial}^{(2)} + H_{\partial}^{(3)} \quad \text{with} \quad [H_{\partial}^{(i)}, H_{\partial}^{(j)}] = 0,$$

then

$$p(R_1, R_2, R_3) \propto \prod_{i=1}^3 d_{R_i} e^{-t_i C_2(R_i)}.$$

*Applied to the assumed transportable refinement-directed edge-sector colimit on the realized branch, together with the rigid symmetric  $C^*$ -tensor and faithful bosonic fiber-functor premises of the compact-gauge theorem, Tannaka–Krein reconstruction first yields some compact group  $G$ . Under the MAR admissibility package, and once the admissible class includes one connected abelian charge factor, the selected connected Lie realization is  $SU(3) \times SU(2) \times U(1)$  up to finite quotient.*

**Corollary A.11** (No leptoquarks, hence no gauge-mediated proton decay). *With product gauge group, the adjoint representation of the full connected gauge group is*

$$(8, 1, 0) \oplus (1, 3, 0) \oplus (1, 1, 0),$$

*equivalently the derived nonabelian adjoint is*

$$(8, 1, 0) \oplus (1, 3, 0).$$

*There are therefore no gauge generators in mixed representations  $(3, 2, \pm 5/6)$ , i.e. no simple-GUT  $X, Y$  bosons on the realized branch. The structural claim is exactly*

$$\tau_p^{(\text{gauge})} = \infty,$$

*meaning that the gauge-boson proton-decay channel is absent. Scalar-mediated or higher-dimensional baryon-violating operators remain outside this structural corollary.*

**Proton-spin continuation benchmark.** On the realized D8–D9 branch the color factor is  $SU(3)$ , so the structural OPH input for proton-spin bookkeeping is

$$C_F = \frac{4}{3}, \quad C_A = 3.$$

If one adds the QCD-dependent continuation ansatz that quark/gluon spin sharing equilibrates according to these Casimirs, then

$$\Delta\Sigma \approx \frac{C_F}{C_F + C_A} = \frac{4}{13} \approx 0.308.$$

Against a representative lattice benchmark  $\Delta\Sigma \approx 0.286$ , this lands within about 8%, but it is only a deferred continuation benchmark. A first-principles OPH derivation of proton spin would require the nonperturbative light-quark/hadron completion plus an explicit map from OPH data to renormalized proton matrix elements.

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