

Reality as a Consensus Protocol: The Fixed-Point Computation That Implements Physics

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Abstract

This paper gives the consensus-theoretic core of Observer-Patch Holography (OPH). If physics is built from local observer descriptions that must agree on overlaps, the first question is whether different repair orders lead to different worlds. On the fixed-cutoff collar branch, the local repair step is not left as an abstract rewrite primitive: it is read from exact Markov splice on exact collars or from declared Petz/Fawzi–Renner reconciliation on recoverable collars, and committed only under the declared local-fit contract on the overlaps it touches. That contract makes the inconsistency potential Φ a Lyapunov functional for the accepted repair moves, and the fixed-cutoff union-collar gluing package makes overlapping repairs parenthesization-independent on the physical quotient. Under repair completeness, the repair dynamics therefore has a unique schedule-independent normal form. The paper also identifies the loop obstructions to global consistency, shows why physical uniqueness belongs on the gauge quotient and its physical observable algebra rather than on raw microscopic representatives, proves observable-level confluence on the quantum lift even when microscopic representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings inside one declared quotient-local glued state, and gives a fixed-point account of stable operator-algebraic records. This is the finite patch-net theorem package that supports the broader relativity, gauge, particle, and observer branches in the OPH suite.

1 Introduction

This paper gives a finite patch-net formulation of the OPH consensus picture. A finite graph of observer patches carries local state spaces and overlap maps. Local recovery-derived repair maps act asynchronously, each modifying only one patch or a bounded neighborhood, and global consistency means agreement on every overlap. The central mathematical problem is whether this repair dynamics admits a unique normal form and how global obstructions to reconciliation are encoded.

The resulting theorem package has two layers. The first layer concerns convergence: the declared local-fit contract on touched overlaps makes Φ a Lyapunov functional for accepted repair moves, so termination is derived on the finite patch net; the fixed-cutoff union-collar gluing package then turns competing local repairs into a local diamond on the physical quotient, so under repair completeness every initial configuration has a unique schedule-independent normal form. The second layer concerns obstructions: pairwise overlap agreement does not ensure a global solution, and the obstruction is holonomic. On the abelian branch it is the cycle sum of edge data; on the genuinely noncentral branch it is a crossed-module Čech class.

Gauge symmetry enters as invariance under changes of hidden local representation that preserve overlap data. When the repair step is read only on that overlap-invariant quotient, the normal-form map descends to the gauge quotient, so physical uniqueness is a quotient statement. On the quantum lift, the same quotient-local carrier determines a unique terminal state on every declared physical observable algebra, even when microscopic representative lifts differ by gauge or sector relabelings inside one quotient-local glued state. The observation layer is carried by finite observer-accessible record algebras generated by central or quantitatively stable approximately commuting projectors. These results form the patch-net formulation used in the broader OPH literature.

This paper proves four core results:

1. **Asynchronous confluence.** For the declared accepted repair law, the local-fit contract makes Φ a Lyapunov functional and hence gives termination, while the fixed-cutoff union-collar gluing package gives the local diamond on the physical quotient; under repair completeness, every initial state has a unique normal form, independent of update schedule (Theorem 3.9).
2. **Cycle obstruction.** For affine overlap constraints over an abelian group, global consistency holds if and only if the holonomy vanishes on every cycle (Theorem 4.1). The parity triangle gives the minimal frustrated example, and Theorem 4.4 extends the same logic to the crossed-module higher-gauge defect hierarchy used later in OPH.
3. **Gauge quotient and observable-level confluence.** When local repair is induced on the overlap-invariant quotient, the normal-form map descends to that quotient, and the induced terminal state on every declared physical observable algebra is unique there even when microscopic representatives differ by gauge or sector relabelings inside one quotient-local glued state (Theorems 5.2, 5.5).
4. **Record algebra and stability.** On the fixed-cutoff observer-accessible surface, central record projectors carry Born/Lüders measurement directly, and approximate record projectors inherit explicit $(\varepsilon, \delta_{\text{rec}})$ stability bounds on the same event surface (Theorem 6.2).

The repair step itself is not a free rewrite primitive here. On the fixed-cutoff collar branch, a local update is obtained from exact Markov splice or from a declared Petz/Fawzi–Renner recovery channel and then read on overlap-invariant physical data. The fixed-point theorem below still isolates the separate remaining burdens cleanly: the declared repair law already includes the touched-overlap local-fit contract that yields Lyapunov descent of Φ on accepted moves, while the fixed-cutoff gluing package supplies the parenthesization-independent union-collar glue used for the local diamond. What remains above that step is repair completeness and, on the Petz branch, the support/CPTP clause stated later in Proposition C.5.

We also define a fitness functional over a finite candidate space of reconciliation laws and prove that replicator dynamics monotonically increases mean fitness (Theorem 7.4). This gives a clean mathematical model for finite-candidate law selection, not a universality theorem or a literal cosmological dynamics claim.

The results here are exact theorems about a computational model. The companion OPH manuscript develops the broader physics branches conditionally and keeps structural theorems, scaling-limit branches, calibration-sector outputs, and phenomenological continuations distinct [1].

2 Patch Nets, Overlaps, and Global Consistency

Definition 2.1 (Patch net). *Let $G = (V, E)$ be a finite connected graph. Each vertex $i \in V$ is an **observer patch** with finite local state space S_i . The global state space is*

$$\Sigma := \prod_{i \in V} S_i.$$

For each edge $e = \{i, j\} \in E$, let I_e be an interface alphabet and let

$$\pi_{i,e} : S_i \rightarrow I_e, \quad \pi_{j,e} : S_j \rightarrow I_e$$

*be the interface projection maps. A global state $s = (s_i)_{i \in V} \in \Sigma$ is **consistent on edge** $e = \{i, j\}$ iff*

$$\pi_{i,e}(s_i) = \pi_{j,e}(s_j).$$

The global consistency set is

$$C := \{s \in \Sigma : \forall e = \{i, j\} \in E, \pi_{i,e}(s_i) = \pi_{j,e}(s_j)\}.$$

For exposition we use a finite pairwise-overlap graph. The hypergraph version is straightforward: replace edges by hyperedges and pairwise equality by a common interface label on each hyperedge. Nothing in the proofs depends on the pairwise restriction.

The picture: each observer holds a local state, and neighboring observers share an interface through which they can compare notes. A universe-state is physically admissible exactly when all neighbors agree on their shared data. This is a constraint satisfaction problem (CSP), and the consistent states are the codewords.

Definition 2.2 (Inconsistency potential). *For each edge e , choose a weight $w_e > 0$ and a function $d_e : I_e \times I_e \rightarrow \mathbb{R}_{\geq 0}$ with $d_e(a, b) = 0 \iff a = b$. On the declared fixed-cutoff branch, d_e is the overlap score used by the local acceptance contract on that interface. Define*

$$\Phi(s) := \sum_{e=\{i,j\} \in E} w_e d_e(\pi_{i,e}(s_i), \pi_{j,e}(s_j)).$$

Then $s \in C \iff \Phi(s) = 0$.

So Φ is the total disagreement energy of the universe. Consistent states have zero energy. Everything else is frustrated.

3 Asynchronous Reconciliation and the Main Fixed-Point Theorem

Definition 3.1 (Recovery-derived local repair law). *Fix for each patch i a finite collar chart $A_i - B_i - D_i$ around the overlaps touched by i , together with a fixed local decoder from repaired collar data back to the finite patch label at i . A **law** λ is a family of local repair maps*

$$T_i^\lambda : \Sigma \rightarrow \Sigma \quad (i \in V)$$

such that T_i^λ changes only the state of patch i (or, more generally, only a bounded neighborhood of i), and the local update is induced by one of the declared OPH recovery moves on that collar, where $\omega_{A_i B_i}(s)$ denotes the $A_i \cup B_i$ marginal of the current collar state encoded by s :

- (i) exact Markov splice on the collar, using Theorem A.1 when $I(A_i : D_i | B_i) = 0$; or
- (ii) a declared recoverability channel

$$(\text{id}_{A_i} \otimes \mathcal{R}_i)(\omega_{A_i B_i}(s)), \quad \mathcal{R}_i = \mathcal{R}_{\sigma_i, \mathcal{N}_i},$$

with \mathcal{R}_i in the Petz/Fawzi–Renner class of Definition C.3 and Theorem A.1.

The decoder back to S_i is bookkeeping for the finite patch presentation; the physical content is the repaired collar state on the declared fixed-cutoff branch. Write $s \rightarrow_i t$ iff $t = T_i^\lambda(s) \neq s$. Let $\rightarrow := \bigcup_{i \in V} \rightarrow_i$, and let \rightarrow^* be its reflexive-transitive closure. A state $s \in \Sigma$ is a **normal form** iff $T_i^\lambda(s) = s$ for all $i \in V$.

Definition 3.2 (Touched-overlap potential and accepted local-fit contract). For each repair site i , let

$$E_i^{\text{touch}} := \{e \in E : \text{the interface data on } e \text{ may change under } T_i^\lambda\}.$$

Define the touched-overlap potential

$$\Phi_i(s) := \sum_{e=\{u,v\} \in E_i^{\text{touch}}} w_e d_e(\pi_{u,e}(s_u), \pi_{v,e}(s_v)).$$

On the declared fixed-cutoff branch, a recovery-derived candidate is committed only if it strictly lowers this touched-overlap score:

$$s \rightarrow_i t \implies \Phi_i(t) < \Phi_i(s).$$

This is the patch-net form of the regulator-side monotone local-fit contract carried by the declared repair package.

Definition 3.3 (Overlap-associative union-collar gluing). Fix $s \in \Sigma$ and two enabled repairs $s \rightarrow_i t$, $s \rightarrow_j u$. Write

$$E_{ij}^{\text{touch}} := E_i^{\text{touch}} \cup E_j^{\text{touch}}.$$

The declared repair branch is **overlap-associative** if the following hold.

- (i) If $E_i^{\text{touch}} \cap E_j^{\text{touch}} = \emptyset$, the two accepted local updates have disjoint support on the declared branch and therefore commute.
- (ii) If $E_i^{\text{touch}} \cap E_j^{\text{touch}} \neq \emptyset$, there is a finite union collar U_{ij} covering the interfaces in E_{ij}^{touch} such that the physical glued state on U_{ij} is parenthesization-independent on the quotient, in the sense of Proposition B.8, and the local decoders/lifts of Definition 3.1 are restriction-compatible on nested collars.

This is the concrete compatibility package used below to complete the local diamond.

Remark 3.4 (Inputs and remaining burdens). The repair step is therefore not an abstract rewrite primitive. Its declared inputs are the fixed-cutoff collar chart, either exact Markov splice or a chosen Petz/Fawzi–Renner recovery channel, a local decoder/lift back to the finite patch presentation, and the touched-overlap local-fit contract of Definition 3.2, together with the support-local disjoint-commutation clause and the restriction-compatible union-collar package of Definition 3.3. The parenthesization-independent quotient-local glue used there is supplied by Proposition B.8 from the fixed-cutoff center-sector / higher-gauge gluing package. What the present theorem package still assumes rather than derives is repair completeness. On the Petz branch, full CPTP action on all inputs also requires the support clause recorded later in Proposition C.5. On broader branches one must also prove that the declared union-collar compatibility is preserved under refinement or branch change.

The theorem package separates the imported repair-law data from the remaining theorem-local inputs cleanly:

Assumption 3.5 (Repair completeness). $s \in C \iff \forall i \in V, T_i^\lambda(s) = s$.

Normal forms are exactly the globally consistent states. The dynamics is neither too weak (missing some inconsistencies) nor too strong (repairing things that were fine).

Proposition 3.6 (Accepted repair moves are Lyapunov-decreasing). *For the accepted repair law of Definitions 3.1 and 3.2, every enabled repair strictly decreases the inconsistency potential:*

$$s \rightarrow t \implies \Phi(t) < \Phi(s).$$

Proof. Write $s \rightarrow_i t$. If $e \notin E_i^{\text{touch}}$, then the interface data on e are unchanged by T_i^λ , so the corresponding term in Φ is the same for s and t . Therefore

$$\Phi(t) - \Phi(s) = \Phi_i(t) - \Phi_i(s) < 0$$

by Definition 3.2. □

Proposition 3.7 (Termination from the OPH Lyapunov functional). *Under Proposition 3.6, every repair sequence is finite; equivalently, the repair relation \rightarrow is terminating.*

Proof. Because each S_i is finite, the global state space $\Sigma = \prod_{i \in V} S_i$ is finite, so the value set $\Phi(\Sigma) \subset \mathbb{R}_{\geq 0}$ is finite as well. Along any nontrivial repair step $s \rightarrow t$, Proposition 3.6 gives $\Phi(t) < \Phi(s)$. An infinite repair sequence would therefore produce an infinite strictly descending chain in the finite ordered set $\Phi(\Sigma)$, which is impossible. □

Proposition 3.8 (Local confluence from overlap-associative gluing). *For the accepted repair law of Definitions 3.1, 3.2, and 3.3, the repair relation is locally confluent.*

Proof. Take $s \rightarrow_i t$ and $s \rightarrow_j u$. If $E_i^{\text{touch}} \cap E_j^{\text{touch}} = \emptyset$, then Definition 3.3(i) gives

$$T_j^\lambda(T_i^\lambda(s)) = T_i^\lambda(T_j^\lambda(s)).$$

So with $v := T_j^\lambda(T_i^\lambda(s))$ we have $t \rightarrow_j v$ and $u \rightarrow_i v$.

Assume now $E_i^{\text{touch}} \cap E_j^{\text{touch}} \neq \emptyset$. Let U_{ij} be the union collar from Definition 3.3(ii), and let $\bar{v}_{ij}(s)$ denote the quotient-local glued state on U_{ij} determined by the exterior marginals of s . Proposition B.8 makes this state independent of whether the local splice/recovery is parenthesized as i then j or j then i . Because the local decoders are restriction-compatible on nested collars, the one-site repairs $s \rightarrow_i t$ and $s \rightarrow_j u$ are precisely the i - and j -restrictions of $\bar{v}_{ij}(s)$. Applying that same restriction-compatible decoder to the unfinished subcollar therefore produces the complementary accepted local step from t and from u into representatives of the same quotient-local union-collar state. Hence there exists $v \in \Sigma$ with $t \rightarrow^* v$ and $u \rightarrow^* v$, where v is any representative of $\bar{v}_{ij}(s)$. □

Theorem 3.9 (Asynchronous confluence / fixed-point law). *For the accepted repair law of Definitions 3.1, 3.2, and 3.3, under Assumption 3.5, every initial state $s \in \Sigma$ has a unique normal form*

$$\text{nf}_\lambda(s) \in C,$$

and every maximal asynchronous repair execution from s terminates at that same state. The terminal state is independent of update order.

Proof. By Proposition 3.7, the repair relation \rightarrow is terminating. By Proposition 3.8, it is locally confluent. Newman’s lemma [2] therefore makes it confluent. A terminating confluent repair relation has a unique normal form reachable from each initial state. By Assumption 3.5, the normal forms are exactly C . Every maximal execution terminates and reaches $\text{nf}_\lambda(s)$, and that terminal state is independent of update order. \square

Corollary 3.10 (Objective law is schedule-independent). *Let $M : \Sigma \rightarrow Y$ be any observable. Under the hypotheses of Theorem 3.9, $M(\text{nf}_\lambda(s))$ is independent of the asynchronous update schedule. If physical law is identified with the map $s \mapsto M(\text{nf}_\lambda(s))$, then physical law is objective.*

Proof. All schedules from the same initial s terminate at $\text{nf}_\lambda(s)$, so all yield the same M -value. \square

At this stage objectivity is identified with schedule-independent convergence of the repair dynamics; Theorem 5.5 sharpens this to schedule-independent convergence of the physical observable algebra even when microscopic representatives are not unique.

4 Why Local Agreement Is Not Enough: Cycle Holonomy and Frustration

Now we show that the story has a twist: just because every pair of neighbors agrees does not mean the whole system is consistent.

Theorem 4.1 (Cycle-obstruction / holonomy criterion). *Let A be an abelian group, and let $G = (V, E)$ be a connected graph with an arbitrary orientation on each edge. For each oriented edge $e : u \rightarrow v$, assign a label $b_e \in A$. Consider the affine consistency equations*

$$x_v - x_u = b_e \quad \text{for every oriented edge } e : u \rightarrow v,$$

where $x_v \in A$ are unknown patch labels. A global solution $x : V \rightarrow A$ exists if and only if for every cycle $C \subseteq G$,

$$\sum_{e \in C} \varepsilon_C(e) b_e = 0,$$

where $\varepsilon_C(e) = +1$ if the cycle traverses e in the chosen orientation and -1 otherwise.

Proof. Necessity. Suppose x is a solution. Summing the edge equations around any cycle C ,

$$\sum_{e:u \rightarrow v \in C} \varepsilon_C(e) (x_v - x_u) = \sum_{e \in C} \varepsilon_C(e) b_e.$$

The left side telescopes to 0 because every vertex appears once with $+$ sign and once with $-$ sign.

Sufficiency. Fix a root $r \in V$. For any vertex v , choose a path $P_{r \rightarrow v}$ and define

$$x_v := \sum_{e \in P_{r \rightarrow v}} \varepsilon_{P_{r \rightarrow v}}(e) b_e, \quad x_r := 0.$$

If P and P' are two paths from r to v , traversing P followed by the reverse of P' yields a cycle. By the vanishing-holonomy assumption, the total signed sum is zero, so x_v is well-defined. For any edge $e : u \rightarrow v$, extending a path to u by that edge gives $x_v = x_u + b_e$. \square

Corollary 4.2 (Parity triangle: pairwise consistency is not enough). *Take $A = \mathbb{Z}_2$ on the triangle A - B - C - A with edge labels $b_{AB} = 0$, $b_{BC} = 0$, $b_{CA} = 1$. Each individual edge equation is satisfiable. But the global system is not: the cycle sum is $0 \oplus 0 \oplus 1 = 1 \neq 0$.*

This example shows that pairwise consistency does not imply a global solution. The obstruction is carried by the cycle.

Corollary 4.3 (Stable defects as frustrated holonomy). *Define the defect energy*

$$\Phi_b(x) := \sum_{e:u \rightarrow v \in E} w_e \mathbf{1}[x_v - x_u \neq b_e].$$

If the cycle-holonomy condition fails, then $\min_{x:V \rightarrow A} \Phi_b(x) > 0$. Every minimizer contains irreducible residual inconsistency.

Proof. If $\min_x \Phi_b(x) = 0$, some assignment satisfies all edge equations, contradicting Theorem 4.1. \square

Residual inconsistencies of this type cannot be removed by local repair moves. In the OPH interpretation they are stable topological defects of the reconciliation dynamics.

Theorem 4.4 (Higher-gauge defect hierarchy). *Let a finite overlap nerve carry crossed-module defect data (g_{ij}, h_{ijk}) for a compact crossed module*

$$H \xrightarrow{\partial} G.$$

Under local rechartings by

$$C^1(N, H) \rtimes C^0(N, G),$$

the nonabelian Čech class

$$q = [(g, h)] \in \check{H}^2(N, H \rightarrow G)$$

is invariant. Strict global reconciliation exists if and only if $q = 0$, and nonzero q labels stable fixed-cutoff higher-gauge defects.

Proof. The allowed rechartings are exactly the crossed-module coboundaries. So they preserve the class and strictify the weak gluing data precisely when the class vanishes. A nonzero class therefore gives a topologically protected residual obstruction, in the full crossed-module defect hierarchy rather than only in the abelian truncation. \square

5 Gauge Symmetry as Implementation Hiding

Definition 5.1 (Gauge action). *Let $\Gamma = \prod_{i \in V} \Gamma_i$ act on Σ componentwise. The action is a **gauge action** if for every $e = \{i, j\} \in E$,*

$$\pi_{i,e}(\gamma_i \cdot x) = \pi_{i,e}(x) \quad \forall x \in S_i, \forall \gamma_i \in \Gamma_i.$$

Gauge changes alter hidden local representations but do not alter overlap data.

Gauge transformations change hidden local representations while leaving overlap data fixed. Write

$$q : \Sigma \rightarrow \Sigma/\Gamma, \quad q(s) = [s],$$

for the gauge-orbit map. A **physical repair law** is the family of quotient-local maps

$$\bar{T}_i^\lambda : \Sigma/\Gamma \rightarrow \Sigma/\Gamma$$

induced by the recovery-derived collar updates of Definition 3.1. A **representative repair family** is any choice of lifts

$$T_i^\lambda : \Sigma \rightarrow \Sigma$$

such that

$$q \circ T_i^\lambda = \bar{T}_i^\lambda \circ q.$$

This is the finite patch-net form of saying that the repair step is defined first on overlap-invariant physical data and only then lifted to hidden representatives. The remaining problem here is therefore not to invent a repair rule, but to establish repair completeness on the declared fixed-cutoff branch and the support/CPTP clause on the Petz branch where that channel is used. The touched-overlap local-fit contract gives Lyapunov Φ -descent on accepted moves, and Proposition B.8 together with Definition 3.3 supplies the quotient-local compatibility package used for the local diamond. Stability of that package under refinement or branch change is a separate question. The point here is also that no extra gauge-covariance axiom is needed once repair is formulated on the quotient.

Theorem 5.2 (Gauge quotient theorem). *Under the gauge action of Definition 5.1, any representative lift of a physical repair law as just defined, and the hypotheses of Theorem 3.9,*

$$q(\text{nf}_\lambda(\gamma \cdot s)) = q(\text{nf}_\lambda(s)) \quad \forall \gamma \in \Gamma, \forall s \in \Sigma.$$

Hence the normal-form map descends to the quotient:

$$\bar{\text{nf}}_\lambda : \Sigma/\Gamma \rightarrow \Sigma/\Gamma, \quad [s] \mapsto [\text{nf}_\lambda(s)].$$

Proof. Suppose $s \rightarrow_i t$, so $t = T_i^\lambda(s) \neq s$. If $s' = \gamma \cdot s$, then

$$q(T_i^\lambda(s')) = \bar{T}_i^\lambda(q(s')) = \bar{T}_i^\lambda(q(s)) = q(T_i^\lambda(s)).$$

Thus gauge-equivalent inputs induce the same repaired orbit, and by induction the orbit reached after any repair sequence depends only on the initial orbit. Every maximal repair sequence from s ends at $\text{nf}_\lambda(s)$ by Theorem 3.9, so the terminal orbit $q(\text{nf}_\lambda(s))$ depends only on $q(s) = [s]$. This makes

$$[s] \longmapsto [q(\text{nf}_\lambda(s))]$$

well-defined on Σ/Γ . □

Corollary 5.3 (Gauge-invariant law). *If $M : \Sigma \rightarrow Y$ is gauge-invariant ($M(\gamma \cdot s) = M(s)$ for all γ), then $M(\text{nf}_\lambda(s))$ depends only on the gauge orbit $[s]$, not the representative.*

Proof. Because M is gauge-invariant, it factors through the orbit map: $M = \bar{M} \circ q$ for some $\bar{M} : \Sigma/\Gamma \rightarrow Y$. Theorem 5.2 gives

$$q(\text{nf}_\lambda(\gamma \cdot s)) = q(\text{nf}_\lambda(s)),$$

hence

$$M(\text{nf}_\lambda(\gamma \cdot s)) = \bar{M}(q(\text{nf}_\lambda(\gamma \cdot s))) = \bar{M}(q(\text{nf}_\lambda(s))) = M(\text{nf}_\lambda(s)).$$

□

Definition 5.4 (Physical observable algebra on the quantum lift). *Fix a finite patch region or union collar R on the declared fixed-cutoff quantum lift of Appendix B. Its **physical observable algebra** $\mathcal{A}_{\text{phys}}(R)$ is the fixed-point collar algebra under the compact boundary redundancy action on the ordinary or central-defect branch, or the corresponding quotient-local algebra on the genuinely noncentral branch. The central sector projectors carried by the collar decomposition belong to $Z(\mathcal{A}_{\text{phys}}(R))$. A **physical observable** is any $X \in \mathcal{A}_{\text{phys}}(R)$. For a microscopic representative $s \in \Sigma$, let ω_R^s denote the induced state on $\mathcal{A}_{\text{phys}}(R)$.*

Theorem 5.5 (Observable-level confluence on the quantum lift). *Under the hypotheses of Theorems 3.9 and 5.2, every initial orbit $[s] \in \Sigma/\Gamma$ has a unique quotient normal form*

$$\overline{\text{nf}}_\lambda([s]) \in q(C).$$

Fix a declared region R . Let $t, u \in \Sigma$ be terminal microscopic representatives reached by representative lifts of maximal repair sequences from initial states in the same orbit $[s]$. If the induced R -collar states of t and u are representatives of the same quotient-local glued state in the sense of Proposition B.8, then for every physical observable $X \in \mathcal{A}_{\text{phys}}(R)$,

$$\omega_R^t(X) = \omega_R^u(X).$$

Hence all physical observables converge to the same overlap-consistent values even when the microscopic terminal representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings inside one declared quotient-local glued state.

Proof. Theorems 3.9 and 5.2 give the unique quotient normal form $\overline{\text{nf}}_\lambda([s]) = [\text{nf}_\lambda(s)] \in q(C)$. Now let t and u be as stated. By Corollary B.10, representatives of the same quotient-local glued state induce the same state on the physical observable algebra $\mathcal{A}_{\text{phys}}(R)$, including the central sector projectors carried by that algebra. Therefore $\omega_R^t(X) = \omega_R^u(X)$ for every $X \in \mathcal{A}_{\text{phys}}(R)$. \square

Remark 5.6 (Inputs and remaining boundary for observable-level confluence). Theorem 5.5 does not add a new repair hypothesis beyond the D1 package. Its inputs are exactly the representative-level confluence theorem, quotient descent of the repair law, Definition 5.4, and Corollary B.10 from the fixed-cutoff union-collar gluing package. What remains open is not fixed-cutoff observable uniqueness on that carrier, but extension of the same statement to broader refinement-stable branches where the declared union-collar compatibility is only approximate or where the chosen physical observable algebra itself changes under refinement.

Corollary 5.7 (Inert ancillary refinement does not change physical law). *Let $K = \prod_i K_i$ be a finite ancillary state space and define $\Sigma^\eta := \Sigma \times K$. Lift the repair maps by*

$$T_i^{\lambda, \eta}(s, k) := (T_i^\lambda(s), k).$$

If $M^\eta : \Sigma^\eta \rightarrow Y$ ignores the ancillary factor,

$$M^\eta(s, k) = M(s),$$

with M gauge-invariant on Σ , then

$$M^\eta(\text{nf}_\lambda^\eta(s, k)) = M(\text{nf}_\lambda(s))$$

for all $(s, k) \in \Sigma^\eta$.

Proof. Because the ancillary factor is inert,

$$\text{nf}_\lambda^\eta(s, k) = (\text{nf}_\lambda(s), k).$$

Therefore $M^\eta(\text{nf}_\lambda^\eta(s, k)) = M(\text{nf}_\lambda(s))$, and Corollary 5.3 supplies gauge-orbit independence. \square

Physical uniqueness therefore holds on the quotient by gauge or implementation hiding. The same statement remains unchanged under inert ancillary stabilization.

6 Record Algebras and the Operator Observation Layer

Inputs used here. From the fixed-cutoff collar package we use only the observer-accessible finite-dimensional algebra on one completed compare/write/verify slice, the declared pointer and overlap-sector projectors read on that same slice, and the trace-distance control on restored accessible states when restoration is invoked. No continuum lift and no broader observer-metaphysical premise is used here.

Definition 6.1 (Exact record algebras and approximate record presentations). *Fix one completed observer-accessible slice at cycle t , and let $\mathcal{A}_t^{\text{acc}}$ be the corresponding finite-dimensional accessible algebra. An exact record presentation is a family of orthogonal projectors $\{\hat{P}_a(t)\}_a \subset Z(\mathcal{A}_t^{\text{acc}})$ whose generated algebra*

$$\mathcal{Z}_{\text{rec}}(t) := \text{Alg}(\{\hat{P}_a(t)\}_a)$$

is finite and commutative.

An approximate record presentation on the same declared readout slots is a family of projectors $\{P_a(t)\}_a \subset \mathcal{A}_t^{\text{acc}}$ together with an exact record presentation $\{\hat{P}_a(t)\}_a$ such that

$$\delta_{\text{rec}}(t) := \max_a \|P_a(t) - \hat{P}_a(t)\|$$

is finite.

Theorem 6.2 (Record algebra, Born-Lüders update, and quantitative stability). *Fix one completed observer-accessible slice at cycle t .*

1. *The exact record projectors generate a finite commutative central algebra $\mathcal{Z}_{\text{rec}}(t) \subset Z(\mathcal{A}_t^{\text{acc}})$.*
2. *For every event E in the finite event algebra generated by $\mathcal{Z}_{\text{rec}}(t)$,*

$$\mathbb{P}_t(E) = \text{Tr}(\rho_t \hat{P}_E(t)), \quad \rho_t|_{E=} = \frac{\hat{P}_E(t) \rho_t \hat{P}_E(t)}{\text{Tr}(\rho_t \hat{P}_E(t))}.$$

3. *If no accepted repair between t and $t + 1$ touches the support of $\hat{P}_E(t)$, then the next read of E has probability 1.*
4. *If $\{P_a(t)\}_a$ is an approximate record presentation with modulus $\delta_{\text{rec}}(t)$, then*

$$\|[P_a(t), P_b(t)]\| \leq 4\delta_{\text{rec}}(t)$$

for all declared record projectors $P_a(t), P_b(t)$. Moreover, for every declared elementary record event a and every restored accessible state $\tilde{\rho}_t$ satisfying

$$\|\tilde{\rho}_t - \rho_t\|_1 \leq \varepsilon,$$

one has

$$\left| \text{Tr}(\tilde{\rho}_t P_a(t)) - \text{Tr}(\rho_t \hat{P}_a(t)) \right| \leq \varepsilon + \delta_{\text{rec}}(t).$$

Proof. Because the exact record projectors lie in the center of $\mathcal{A}_t^{\text{acc}}$, they generate a finite commutative central algebra. Finite-dimensional projective measurement on that commuting algebra gives the Born trace and the Lüders conditioned state, proving (1) and (2).

If no accepted repair touches the support of $\hat{P}_E(t)$, then the next completed read is performed on the same central projector surface. After conditioning on E , the state lies in the range of $\hat{P}_E(t)$, so the next read of the same event has probability 1. This gives (3).

For (4), write

$$[P_a(t), P_b(t)] = [P_a(t) - \hat{P}_a(t), P_b(t)] + [\hat{P}_a(t), P_b(t) - \hat{P}_b(t)],$$

because $[\hat{P}_a(t), \hat{P}_b(t)] = 0$. Since every projector has operator norm at most 1,

$$\|[P_a(t), P_b(t)]\| \leq 2\|P_a(t) - \hat{P}_a(t)\| + 2\|P_b(t) - \hat{P}_b(t)\| \leq 4\delta_{\text{rec}}(t).$$

For the probability bound,

$$\left| \text{Tr}(\tilde{\rho}_t P_a(t)) - \text{Tr}(\rho_t \hat{P}_a(t)) \right| \leq \left| \text{Tr}((\tilde{\rho}_t - \rho_t) P_a(t)) \right| + \left| \text{Tr}(\rho_t (P_a(t) - \hat{P}_a(t))) \right|.$$

The first term is bounded by $\|\tilde{\rho}_t - \rho_t\|_1 \|P_a(t)\| \leq \varepsilon$, and the second by $\|\rho_t\|_1 \|P_a(t) - \hat{P}_a(t)\| \leq \delta_{\text{rec}}(t)$. Hence the total error is at most $\varepsilon + \delta_{\text{rec}}(t)$. \square

Merge boundary. What is closed here is the fixed-cutoff operator-algebraic observation surface: a central record algebra on the exact readout slice, Born/Lüders measurement on its event projectors, exact repeated-read stability under the untouched-support hypothesis, and explicit $(\varepsilon, \delta_{\text{rec}})$ control when practical readout projectors are only close to that central surface. What remains beyond this theorem is not another finite-cutoff record lemma but the broader export problem: proving on richer branches that physically relevant pointer surfaces stay close to one such central record algebra and that the same control survives refinement and continuation.

7 Law-Space Selection and Observer Emergence

This section studies a simple meta-selection model on law space. The aim is to formalize one criterion for favoring schedule-robust, observer-supporting, and simple laws; the replicator dynamics below is part of the model, not a claim about literal cosmological dynamics.

We begin by defining what it means for a law to support observers. An observer is treated operationally as a **persistent predictive module**: a subgraph that maintains a stable record algebra and uses its output law to predict its boundary's future behavior.

Definition 7.1 (Schedule robustness). *Fix distributions μ over initial conditions and ν over asynchronous schedules, and a gauge-invariant observable M . For a law λ , define*

$$\mathcal{R}_M(\lambda) := \Pr_{s \sim \mu, \sigma, \tau \sim \nu} [M(\text{nf}_\lambda^\sigma(s)) = M(\text{nf}_\lambda^\tau(s))].$$

If Theorem 3.9 holds for λ , then $\mathcal{R}_M(\lambda) = 1$.

Definition 7.2 (Observer yield). *Let X_t^λ denote the stationary process obtained by repeated local perturbation plus reconciliation under law λ . For each subgraph $U \subseteq V$, let $\mathcal{Z}_U^{\text{rec}}(t)$ be the declared exact record algebra on its observer-accessible surface, or the reference exact algebra when only*

an approximate record presentation is available, and let $Y_U(t)$ be the corresponding finite outcome variable induced by that record algebra. Then U is (η, ε, h) -observer-like if it is record-stable:

$$d_{\text{TV}}(\text{Law}(Y_U(t+h)), \text{Law}(Y_U(t))) \leq \eta,$$

and predictive:

$$I(Y_U(t); X_{\partial U, t+1:t+h}^\lambda) \geq \varepsilon.$$

Define

$$\mathcal{O}_{\eta, \varepsilon, h}(\lambda) := \mathbb{E}[\#\{U \subseteq V : U \text{ is } (\eta, \varepsilon, h)\text{-observer-like}\}].$$

Definition 7.3 (Law fitness). Let $K(\lambda)$ be a description-length penalty. Define

$$f(\lambda) = \alpha \mathcal{R}_M(\lambda) + \beta \mathcal{O}_{\eta, \varepsilon, h}(\lambda) - \gamma K(\lambda),$$

with $\alpha, \beta, \gamma > 0$.

Theorem 7.4 (Replicator monotonicity on law space). Let $\Lambda = \{\lambda_1, \dots, \lambda_m\}$ be candidate laws with population weights $x_i(t)$ under replicator dynamics:

$$\dot{x}_i = x_i(f_i - \bar{f}), \quad f_i := f(\lambda_i), \quad \bar{f} := \sum_{j=1}^m x_j f_j.$$

Then

$$\frac{d}{dt} \bar{f} = \text{Var}_x(f) \geq 0.$$

Mean fitness is nondecreasing, and strictly increasing unless all extant laws have the same fitness.

Proof. Direct computation:

$$\frac{d}{dt} \bar{f} = \sum_i \dot{x}_i f_i = \sum_i x_i (f_i - \bar{f}) f_i = \sum_i x_i f_i^2 - \bar{f}^2 = \text{Var}_x(f) \geq 0.$$

Equality iff all f_i on the support of x are equal. □

This theorem records the monotonicity property of the meta-selection model.

8 Connection to Observer-Patch Holography

The formalism above is the computational skeleton of Observer-Patch Holography (OPH). The observer patches carry von Neumann algebras on a holographic screen S^2 , the overlap projections are restrictions to shared subalgebras, and the consistency condition is algebraic state agreement on overlaps.

The bridge to physics works as follows in the broader paper stack:

- The patch net becomes a net of subregion algebras on S^2 .
- Overlap Consistency, one of the canonical OPH axioms, is the algebraic version of Definition 2.1.
- The recoverability clause of the canonical Recoverable Generalized Entropy axiom provides the collar factorization $\rho_{ABD} = \bigoplus_\alpha p_\alpha \rho_{Ab_L}^{(\alpha)} \otimes \rho_{b_R D}^{(\alpha)}$ and the declared Petz/Fawzi–Renner recovery channels from which the local repair moves are built (see Appendix A and Definition C.3).

- Gauge symmetry as implementation hiding (Theorem 5.2) becomes the edge-sector fusion structure from which OPH conditionally reconstructs a compact gauge group and, under the MAR admissibility package, selects the Standard Model gauge group $SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$.
- Stable defects (Corollary 4.3) become the topologically protected excitations that OPH identifies with particles.
- The record-algebra theorem (Theorem 6.2) provides the formal basis for the fixed-cutoff observation layer in OPH, where records are carried by central or quantitatively stable approximately commuting projectors in overlap centers.

The companion manuscripts develop a conditional gravity branch from entanglement equilibrium and modular geometry, the SM gauge-group closure from edge-sector admissibility plus MAR, and a worldsheet/string reorganization from large- N heat-kernel asymptotics. Input-dependent cosmological statements and later phenomenological continuations are kept explicitly separate from that recovered-core claim set.

This paper provides the finite patch-net foundation for the broader companion suite. When that suite is described using labels such as $D3-D5$ or $D7-D9$, those are internal documentary node labels from the companion derivation ledger [1]; they are not external references, and that companion ledger is the place to look up the definitions.

9 Discussion and Open Problems

This paper proves the fixed-point consensus spine of OPH: schedule-independent normal forms, holonomy obstructions, gauge-quotient invariance, and the fixed-cutoff operator-record theorem. It also gives a clean law-selection meta-model. The conditional relativity chain and the realized Standard Model structural chain remain the recovered core. The capacity relation is separate and input-dependent. Downstream phenomenology requires additional premises beyond the consensus results proved here.

Complexity status. On a fixed finite patch net, the accepted reconciliation dynamics is a finite-state asynchronous rewrite system on Σ . Under Theorem 3.9, every accepted repair run has at most $|\Phi(\Sigma)| - 1 \leq |\Sigma| - 1$ nontrivial steps, because each accepted move strictly lowers Φ and the value set $\Phi(\Sigma)$ is finite. Exact normal-form computation is therefore decidable by direct iteration of accepted local repairs. What this paper does *not* prove is a uniform polynomial-time bound, a sharper complexity-class placement for families of growing patch nets, or any hardness lower bound.

Approximate-stability status. The theorem-grade consensus statement is exact on the declared fixed-cutoff branch. Approximate control is presently collar-local: Theorem A.1 gives the exact-Markov modulus $\delta_{A:B:D}^M(\varepsilon) \rightarrow 0$ on one fixed finite-dimensional collar model and the one-shot recovery comparison bound $2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}$, while Theorem 6.2 gives the $(\varepsilon, \delta_{\text{rec}})$ repeated-read stability bound for approximate record projectors. This paper does *not* prove a global noisy-consensus theorem asserting that arbitrarily long asynchronous approximate repair sequences converge to a unique approximate quotient normal form with uniformly controlled accumulated error.

Expressive-power status. The law-selection model of Theorem 7.4 is a finite-candidate monotonicity result, not a universality theorem. For each fixed patch net the theorem package proves a finite-state exact reconciliation mechanism rather than an unbounded universal machine. A

genuine universality claim would require an explicit uniform family of patch nets and repair laws that simulates arbitrary circuits or machines with stated encoding overhead and robustness under asynchronous schedules. No such theorem is supplied here.

With that status split fixed, six explicit downstream directions remain:

1. **Coarse-graining.** When does reconciliation commute with renormalization? If you coarse-grain the patch net and then reconcile, do you get the same result as reconciling first and then coarse-graining? The conditions under which these operations commute would connect the discrete model to continuum field theory.
2. **Defect classification and refinement-limit transportability.** The fixed-cutoff hierarchy extends from abelian frustrations to crossed-module classes $q \in \check{H}^2(N, H \rightarrow G)$. The remaining task is to connect those higher-gauge defect sectors to the refinement-stable transportable sector category used in the broader compact-gauge reconstruction lane.
3. **Observable-level confluence beyond the declared fixed-cutoff physical algebra.** Theorem 5.5 closes the fixed-cutoff quantum-lift statement when microscopic representatives differ by gauge relabelings globally or by sector/higher-gauge relabelings on the same declared quotient-local glued state. The remaining question is whether an analogous observable theorem survives on broader refinement-stable branches where the union-collar compatibility is only approximate or where the physical observable algebra itself changes under refinement.
4. **Global approximate-consensus stability.** Theorem A.1 and Theorem 6.2 supply the collar-local perturbative controls carried by this paper. What is absent is a multi-step theorem showing that approximate recovery moves are repair-complete for overlap consistency, preserve support-local disjoint commutation and nested-collar restriction compatibility, satisfy the support/CPTP clause on the Petz branch where needed, and converge under long asynchronous schedules to a unique approximate quotient normal form with controlled accumulated error.
5. **Expressive power / universality.** A universality claim would require an explicit uniform family of patch nets and repair laws that simulates arbitrary circuits or machines with stated encoding overhead and robustness under asynchronous schedules. This paper supplies no such construction, so universality remains an external possibility rather than a theorem-grade output.
6. **Scaling-limit bridge to the downstream gravity/gauge stack.** Under what additional regularity conditions does the consensus protocol emit the prime geometric cap pair and support ordered cut-pair rigidity on that extracted limit, so that the D3–D5 gravity chain and the D7–D9 gauge/matter chain become available? This is also the point at which the broader Phase-II and Phase-III branches must remain disciplined: dark-sector, baryogenesis, spectroscopy, and string/worldsheet topics require extra premises beyond what this paper itself proves.

The last item is the bridge to the downstream gravity/gauge branches. Failure of that bridge would leave the fixed-point theorems proved here unchanged while revising the downstream gravity, gauge, or continuation sectors.

9.1 Conditional BFT and QECC Extensions

The consensus formalism of OPH has natural analogies to classical and quantum distributed Byzantine agreement. Observer patches correspond to protocol nodes, overlap repair corresponds to a quorum vote, and the repair fixed-point corresponds to a consensus state. Under explicit structural assumptions (quorum size $\geq 2f + 1$, partial synchrony, one-vote-per-view, certificate semantics, and DLS-style view-change), a QBFT-style interpretation of OPH repair satisfies safety and liveness (Appendix C, Theorem C.2). On the fixed-cutoff collar branch used here, the repair map is written in exact-splice / Petz form; what remains conditional is the CPTP property on all inputs, which requires either full-rank $\mathcal{N}(\sigma)$ or an explicit domain restriction, and trace-preserving completion is not automatic when $\mathcal{N}(\sigma)$ has a non-trivial kernel (Proposition C.5). A quantum error-correcting interpretation is possible under additional topological structure, but the code-distance / min-cut equality requires explicit conditions on logical-operator homology and boundary geometry (Claim C.9). All of these extensions are conditional or conjectural and are not part of the core theorem package of Paper 4.

A Quantum/Algebraic Lift: Markov-Collar Splice Theorem

This appendix records the algebraic splice statement used to relate the finite patch-net model to the OPH collar formalism.

An observer is written as

$$O = (P, \mathcal{A}(P), \rho, R),$$

where P is the screen patch, $\mathcal{A}(P)$ the local von Neumann algebra, ρ the local state, and R the record algebra.

Theorem A.1 (Markov-collar splice theorem, exact and controlled). *Suppose a collar tripartition A - B - D has exact Markov decomposition*

$$\rho_{ABD} = \bigoplus_{\alpha} p_{\alpha} \rho_{Ab_L^{\alpha}}^{(\alpha)} \otimes \rho_{b_R^{\alpha}D}^{(\alpha)}.$$

Let $\sigma_{b_R^{\alpha}D}^{(\alpha)}$ be any family of normalized environment states compatible with the same right-boundary sectors. Define

$$\rho'_{ABD'} = \bigoplus_{\alpha} p_{\alpha} \rho_{Ab_L^{\alpha}}^{(\alpha)} \otimes \sigma_{b_R^{\alpha}D'}^{(\alpha)}.$$

Then for every observable X supported on $A \cup b_L$,

$$\mathrm{Tr}(X \rho'_{ABD'}) = \mathrm{Tr}(X \rho_{ABD}).$$

Now fix one finite-dimensional collar model and let

$$\mathfrak{M}_{A:B:D} := \{ \tau_{ABD} : I(A : D | B)_{\tau} = 0 \},$$

with exact-Markov distance modulus

$$\delta_{A:B:D}^M(\varepsilon) := \sup \left\{ \inf_{\tau \in \mathfrak{M}_{A:B:D}} \|\omega - \tau\|_1 : I(A : D | B)_{\omega} \leq \varepsilon \right\}.$$

Then

$$\delta_{A:B:D}^M(\varepsilon) \rightarrow 0 \quad (\varepsilon \downarrow 0).$$

Hence if $I(A : D | B)_\omega \leq \varepsilon$ and $\tilde{\omega}_\varepsilon \in \mathfrak{M}_{A:B:D}$ is chosen so that

$$\|\omega - \tilde{\omega}_\varepsilon\|_1 \leq \delta_{A:B:D}^M(\varepsilon),$$

the corresponding exact splice $\tilde{\omega}'_\varepsilon$ satisfies

$$|\mathrm{Tr}(X\omega) - \mathrm{Tr}(X\tilde{\omega}'_\varepsilon)| \leq \|X\|_\infty \delta_{A:B:D}^M(\varepsilon)$$

for every observable X supported on $A \cup b_L$.

Independently, if $I(A : D | B)_\omega \leq \varepsilon$, then there exists a recovery map $\mathcal{R}_{B \rightarrow BD}$ such that

$$\|\omega_{ABD} - (\mathrm{id}_A \otimes \mathcal{R}_{B \rightarrow BD})(\omega_{AB})\|_1 \leq 2\sqrt{1 - e^{-\varepsilon}} \leq 2\sqrt{\varepsilon}.$$

Proof. The exact splice statement is the usual blockwise factorization argument:

$$\mathrm{Tr}(X\rho'_{ABD'}) = \sum_\alpha p_\alpha \mathrm{Tr}\left(X \rho_{Ab_L}^{(\alpha)}\right) \mathrm{Tr}\left(\sigma_{b_R D'}^{(\alpha)}\right).$$

Each right factor is normalized, so the value agrees with the same computation for ρ_{ABD} .

For the controlled statement, compactness of the fixed finite-dimensional state space and continuity of conditional mutual information imply $\delta_{A:B:D}^M(\varepsilon) \rightarrow 0$: otherwise one could find a sequence with $I(A : D | B) \rightarrow 0$ staying a fixed trace distance away from every exact Markov state, contradicting convergence of a subsequence to an exact Markov limit point. Once $\tilde{\omega}_\varepsilon$ is chosen, the exact splice identity for $\tilde{\omega}_\varepsilon$ gives

$$|\mathrm{Tr}(X\omega) - \mathrm{Tr}(X\tilde{\omega}'_\varepsilon)| = |\mathrm{Tr}[X(\omega - \tilde{\omega}_\varepsilon)]| \leq \|X\|_\infty \delta_{A:B:D}^M(\varepsilon).$$

The final inequality is the standard Fawzi–Renner recovery bound [15]. □

This appendix therefore uses exact splice identities in only two regimes: literal exact Markovity, or a controlled collar family on one fixed finite-dimensional model for which $\delta_{A:B:D}^M(\varepsilon) \rightarrow 0$. Small one-shot conditional mutual information is not silently upgraded to an exact normal form.

B Fixed-Cutoff Realization, Quotient Repair, and Edge Centers

This appendix rehouses the fixed-cutoff realization and edge-center items that had been living in the former standalone technical supplement. They belong here because they sharpen the quotient-first repair interpretation used throughout the consensus paper and make the collar boundary data explicit at the same finite patch-net level. On the declared fixed-cutoff collar branch, the local repair step is read from exact Markov splice or a declared Petz/Fawzi–Renner recovery move on that same collar data; representative repair maps are only lifts of the resulting quotient-local update.

B.1 Quotient Repair and UV Underdetermination

At fixed cutoff, each regulator cell x carries a finite-dimensional factor \mathfrak{h}_x , patch algebras are finite type-I algebras, and gauge-as-gluing is realized as a compact boundary redundancy action on cut data. The physical repair law therefore belongs on the overlap-invariant quotient rather than on hidden representatives. If $q : \Sigma \rightarrow \Sigma/\Gamma$ is the quotient by boundary redundancy and \bar{T}_i is the physical quotient update, a representative-level map T_i is only required to be a lift satisfying

$$q \circ T_i = \bar{T}_i \circ q.$$

Hence

$$q(T_i(\gamma \cdot s)) = q(T_i(s))$$

for gauge-equivalent inputs. Quotient descent is therefore structural, while strict representative-level covariance is only implementation bookkeeping. The remaining burden is not to postulate a repair rule, but to prove that the accepted recovery-derived local moves satisfy the stated repair-completeness, support-local disjoint-commutation, nested-collar restriction-compatibility, and Petz-domain control clauses on the declared branch. The touched-overlap acceptance contract is already what yields finite Lyapunov descent and derived termination for accepted moves, while the fixed-cutoff gluing package carries the parenthesization-invariant union-collar state used for the local diamond.

Proposition B.1 (Ancilla-stable UV underdetermination). *Let a fixed-cutoff OPH realization be stabilized by finite ancillary factors K_P in a fixed product state, with observable patch algebras embedded as $\mathcal{A}(P) \otimes \mathbf{1}_{K_P}$ and repair dynamics acting trivially on the ancillas. Then observable expectations on the physical subalgebras, overlap data, the local-Gibbs branch, the collar conditional mutual information $I(A : D \mid B)$, the Fawzi–Renner remainder, the collar Markov modulus, and the quotient normal form are unchanged. Thus the fixed-cutoff theorem package determines the UV branch only modulo such ancillary stabilization together with gauge or implementation hiding, not a unique microscopic presentation.*

Proof. Product ancillas leave physical observables unchanged, cancel additively inside conditional mutual information, and are inert under the repair maps. Hence every invariant listed above is unchanged. \square

B.2 Derived Boundary Data and Ordinary EC

Proposition B.2 (Derived boundary gluing datum). *Choose a finite regulator chart for the patches meeting along a connected cut Σ . Because the local overlap algebras are finite-dimensional matrix algebras, any overlap-consistent recharting is an inner automorphism and is implemented by a unitary on the cut Hilbert space. The compact closure of the subgroup generated by these recharting unitaries is a compact boundary redundancy group K_Σ . If triple-overlap defects are central, the projective composition law lifts to a compact central extension \widehat{K}_Σ ; on the ordinary branch one simply sets $\widehat{K}_\Sigma = K_\Sigma$. A genuinely noncentral 2-group defect is the only obstruction to reducing the overlap transition system to an ordinary compact group action.*

Theorem B.3 (Derived EC decomposition). *Under the fixed-cutoff regulator realization above, and on the ordinary or central-defect branch, the collar Hilbert space is*

$$\mathcal{H}_{B_\delta} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\widehat{K}_\Sigma} \cong \bigoplus_{\alpha} \left(\mathcal{H}_{b_L^\alpha} \otimes \mathcal{H}_{b_R^\alpha} \right),$$

and the center of the collar algebra is generated by the block projectors:

$$Z(\mathcal{A}(B_\delta)) = \bigoplus_{\alpha} \mathbb{C} \cdot \mathbf{1}_\alpha.$$

The right half-collar carries the contragredient representation because it sees inverse transport across the same cut.

Remark B.4. This is the finite-patch-net origin of the collar center used by the later Markov, record, and observer packages. Exact Markovity is still an additional state hypothesis; EC provides the kinematic block structure.

B.3 Higher-Gauge Replacement on the Genuinely Noncentral Branch

Proposition B.5 (Derived higher-gauge cut datum). *On the genuinely noncentral branch, weak overlap gluing on a connected cut Σ is encoded by a compact crossed module*

$$\mathbb{K}_\Sigma = (H_\Sigma \xrightarrow{\partial_\Sigma} G_\Sigma, \triangleright)$$

with defect class

$$q_\Sigma \in \check{H}^2(N_\Sigma, H_\Sigma \rightarrow G_\Sigma),$$

and compact higher-gauge change system

$$\mathcal{T}_\Sigma = C^1(N_\Sigma, H_\Sigma) \rtimes C^0(N_\Sigma, G_\Sigma).$$

Theorem B.6 (Higher-gauge EC decomposition and defect transport). *On the genuinely noncentral branch,*

$$\mathcal{H}_{B_\delta}^{2g} = (\tilde{\mathcal{H}}_{B_L} \otimes \tilde{\mathcal{H}}_{B_R})^{\mathcal{T}_\Sigma} \cong \bigoplus_\lambda (\mathcal{H}_{b_L^\lambda} \otimes \mathcal{H}_{b_R^\lambda}),$$

and

$$Z(\mathcal{A}_{2g}(B_\delta)) = \bigoplus_\lambda \mathbb{C} \cdot \mathbf{1}_\lambda.$$

Moreover the higher-gauge defect class q_Σ is invariant under local rechartings, vanishes iff the defect is removable, and classifies fixed-cutoff genuinely noncentral sectors.

Corollary B.7 (Exact Markov adds the state factorization). *On either the ordinary/central branch of the previous theorem or the genuinely noncentral higher-gauge branch, if in addition*

$$I_\omega(A_\delta : D_\delta \mid B_\delta) = 0,$$

or one passes to the explicitly stated idealized recoverability limit that reduces to exact Markovity, then

$$\rho_{A_\delta B_\delta D_\delta} = \bigoplus_\alpha p_\alpha \left(\rho_{A_\delta b_L^\alpha} \otimes \rho_{b_R^\alpha D_\delta} \right).$$

EC therefore gives the kinematic block decomposition, while exact Markovity is the extra state input that gives the HJPW normal form.

Proposition B.8 (Parenthesization-invariant union-collar gluing). *Fix a finite union collar U built from two overlapping local repair collars on the declared fixed-cutoff branch. On the ordinary or central-defect branch, the physical glued state on U determined by the exterior marginals and sector data is independent of how the local gluings are parenthesized, up to the boundary-redundancy action. On the genuinely noncentral branch, the same statement holds up to the crossed-module change system \mathcal{T}_Σ . Hence the physical quotient-local glued state on U is well defined independently of parenthesization.*

Proof. On the ordinary or central-defect branch, the preceding edge-center decomposition writes the collar algebra as a direct sum of sector blocks generated by central projectors. Reparenthesizing two overlapping local gluings changes only the representative by the central boundary action and cannot move the state between those block projectors, so the quotient-local glued state is unchanged. On the genuinely noncentral branch, the failure of strict composition is recorded by the crossed-module associator, and the higher-gauge theorem above identifies rechartings exactly with the \mathcal{T}_Σ -coboundary action. Reparenthesization therefore changes only the representative inside one \mathcal{T}_Σ -orbit. In either case the physical quotient state is parenthesization-independent. \square

Remark B.9. Approximate recoverability gives controlled deviations from this normal form; it is not implied by EC alone. This is the fixed-cutoff topological package behind the consensus paper’s quotient and record-language surface.

Corollary B.10 (Physical observables are invariant on one quotient-local glued state). *Let U be a finite union collar on the declared fixed-cutoff branch, and let ω_U, ω'_U be two microscopic representatives of the same quotient-local glued state from Proposition B.8. Then every physical observable X on the collar fixed-point / quotient-local algebra has the same expectation in both representatives:*

$$\text{Tr}(X\omega_U) = \text{Tr}(X\omega'_U).$$

In particular the same holds for the central sector projectors and for any observer-accessible record observable generated from them on that same declared surface.

Proof. On the ordinary or central-defect branch, Proposition B.8 says the two representatives differ only by the boundary-redundancy action inside one fixed sector block. The fixed-point collar algebra and its central block projectors are invariant under that action, so their expectation values agree. On the genuinely noncentral branch, the same proposition says the two representatives differ only inside one \mathcal{T}_Σ -orbit, and the quotient-local physical algebra $\mathcal{A}_{\text{phys}}(U)$ of Definition 5.4 is defined precisely on that orbit space. Therefore the induced physical state and all expectations of physical observables agree there as well. \square

C Conditional Distributed-Systems and QECC Extensions of the Consensus Formalism

Honesty labels.

[**Established**] Follows from cited prior work or a complete argument given here.

[**Conditional**] True under additional assumptions not yet derived from OPH first principles.

[**Conjecture**] A plausible open direction, not a settled result.

B.1 Theorem 1 — QBFT Safety Bound

Definition C.1 (QBFT-style protocol). *A consensus protocol is QBFT-style in this analysis if it satisfies the following three structural properties. The safety proof of Theorem C.2 uses all three; the theorem does not hold for protocols lacking any of them without a compensating change to the argument.*

(P1) **One-vote-per-view.** *Each honest node casts at most one vote per view number. A node that has already voted in view v ignores any later request to vote in view v .*

(P2) **Certificate semantics.** *A decision requires a valid quorum certificate: $2f + 1$ distinct, unforgeable, authenticated votes for the same value in the same view.*

(P3) **DLS-style view-change.** *If no certificate is produced within a timeout, every honest node increments the view number by one and a new leader is selected by a fixed deterministic rule. At GST, timeouts fire correctly and the view-change terminates in bounded rounds.*

The Istanbul BFT / QBFT protocol family [6, 7] satisfies (P1)–(P3) and is the intended instance.

Assumptions A1–A6.

- (A1) **Partial synchrony (DLS).** Fixed but initially unknown bounds Δ (message delay) and Φ (processing rates). *Safety* holds unconditionally; *liveness* holds after the Global Stabilisation Time (GST).
- (B2) **Byzantine fault model.** At most f observers behave arbitrarily; the remaining $n - f$ are honest.
- (C3) **Optimal fault bound.** $n \geq 3f + 1$ (necessary: [4]; sufficient: [4]).
- (D4) **Strong quorum connectivity.** Every quorum Q with $|Q| = 2f + 1$ is strongly connected within G : for any $u, v \in Q$ there is a directed path in G contained entirely in Q . This is strictly stronger than requiring the overlap graph of quorums to be connected, and is needed to propagate signed votes within a quorum.
- (E5) **Message authentication.** All messages carry unforgeable digital signatures.
- (F6) **OPH quorum overlap.** Any two quorums Q_a, Q_b of size $2f + 1$ satisfy $|Q_a \cap Q_b| \geq f + 1$ (guaranteed by (A3)).

Theorem C.2 (QBFT Safety Bound [Established, conditional on A1–A6]). *Under assumptions (A1)–(A6), any consensus protocol satisfying (P1)–(P3) of Definition C.1 and run over the OPH observer graph satisfies:*

- (i) **Safety.** *No two honest observers finalise conflicting patch states.*
- (ii) **Liveness.** *After GST, every honest observer finalises within $O(f \cdot \Delta)$ wall-clock time.*
- (iii) **Optimality.** *The bound $f < n/3$ is tight.*

Proof sketch. Safety. Suppose O_a and O_b finalise $s_a \neq s_b$ in the same view. By (P2), each required a certificate of $q = 2f + 1$ votes: sets Q_a, Q_b . By (A3): $|Q_a \cap Q_b| \geq (2f + 1) + (2f + 1) - (3f + 1) = f + 1$. By (A2), at most f are Byzantine, so $Q_a \cap Q_b$ contains an honest O^* . By (A4), O^* 's signed vote is path-reachable within both quorums. By (P1), O^* voted for at most one value — contradiction.

Liveness and Optimality follow from [5] (Thm. 4.4) and [4], cited directly.

Note on FLP. Fischer, Lynch, Paterson [3] is an impossibility result for fully asynchronous systems; it does not bear on achievability under partial synchrony (A1). \square

B.2 Theorem 2 — Convergence of the OPH Repair Map

Definition C.3 (OPH Repair Map — Petz form). *Let $\sigma \in \mathcal{D}(\mathcal{H})$ be a full-rank reference state and $\mathcal{N} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ a quantum channel. The OPH repair map is*

$$\mathcal{R}_{\sigma, \mathcal{N}}(\rho) := \sigma^{1/2} \mathcal{N}^\dagger(\mathcal{N}(\sigma)^{-1/2} \rho \mathcal{N}(\sigma)^{-1/2}) \sigma^{1/2},$$

where \mathcal{N}^\dagger is the adjoint channel and inverses are taken on $\text{supp}(\mathcal{N}(\sigma))$.

Remark C.4 (Petz map vs. trace-distance projection). The closest-point trace-distance projection $\mathcal{P}_S(\rho) := \arg \min_{\tau \in S} \frac{1}{2} \|\rho - \tau\|_1$ is a different object from the Petz map: it is defined by a variational problem in trace-norm geometry and is not CPTP in general. The two coincide only in very special cases not automatic in the OPH setting. All subsequent properties refer exclusively to Definition C.3.

Proposition C.5 (Petz map CPTP — domain-restricted statement [Established, subject to domain restriction]). *Let σ have full support on \mathcal{H} .*

- (a) $\mathcal{R}_{\sigma, \mathcal{N}}$ is completely positive.
- (b) $\mathcal{R}_{\sigma, \mathcal{N}}$ is trace-preserving on $\text{supp}(\mathcal{N}(\sigma))$, i.e., on inputs ρ for which $\mathcal{N}(\sigma)^{-1/2} \rho \mathcal{N}(\sigma)^{-1/2}$ is well-defined.
- (c) If additionally $\mathcal{N}(\sigma)$ has full rank on \mathcal{K} , then $\mathcal{R}_{\sigma, \mathcal{N}}$ is CPTP on all of $\mathcal{B}(\mathcal{K})$.

If $\mathcal{N}(\sigma)$ is not full rank on \mathcal{K} , then either (i) the domain must be restricted to $\text{supp}(\mathcal{N}(\sigma))$, or (ii) pseudoinverses must replace the inverses (generalised Petz map; cf. [10]), or (iii) a regularisation $\mathcal{N}(\sigma) \mapsto \mathcal{N}(\sigma) + \varepsilon \mathbf{1}$ must be introduced. Note that full-rank σ does not prevent $\mathcal{N}(\sigma)$ from being rank-deficient: the channel may map the support of σ into a strict subspace of \mathcal{K} . In the OPH setting, whether $\mathcal{N}(\sigma)$ is full rank depends on the specific overlap channel and must be verified separately (open issue #62).

Proof. Complete positivity follows from composing three CP operations:

- (i) sandwiching by $\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}$ on $\text{supp}(\mathcal{N}(\sigma))$;
- (ii) \mathcal{N}^\dagger ;
- (iii) sandwiching by $\sigma^{1/2}(\cdot)\sigma^{1/2}$.

Trace preservation in the full-rank case: Petz [8]; Fagnola–Umanità [9]. □

Proposition C.6 (Contraction [Conditional]). *Suppose \mathcal{N} is strictly contractive with spectral gap $\lambda \in (0, 1)$:*

$$\sup_{\rho \neq \tau} \frac{\|\mathcal{N}(\rho) - \mathcal{N}(\tau)\|_1}{\|\rho - \tau\|_1} \leq \lambda < 1.$$

Then $\mathcal{R}_{\sigma, \mathcal{N}} \circ \mathcal{N}$ is contractive with coefficient depending on λ and $\text{spec}(\sigma)$. Establishing $\lambda < 1$ for the OPH overlap channel requires analysis of the OPH Hamiltonian (open issue #62).

Conjecture C.7 (Spectral gap [Conjecture]). *The transfer operator \mathcal{T} for iterated application of $\mathcal{R}_{\sigma, \mathcal{N}}$ has a positive spectral gap $\delta > 0$, implying exponential convergence. Proof from OPH first principles is open issue #63.*

Theorem C.8 (Exponential Convergence [Conditional on Conjecture C.7]). *Assuming Conjecture C.7 with gap $\delta > 0$,*

$$\frac{1}{2} \|\mathcal{R}_{\sigma, \mathcal{N}}^{\text{ot}}(\rho) - \sigma\|_1 \leq C e^{-\delta t}$$

for $C > 0$ depending on ρ and \mathcal{N} .

B.3 Theorem 3 — QECC Correspondence

Notation. $N = \dim(\mathcal{H}) = 2^n$ for n physical qubits. Standard notation: $[[n, k, d]]$ stabilizer code; $K = 2^k$; quantum Singleton bound: $k \leq n - 2(d - 1)$.

Claim C.9 (Code distance and min-cut [Conditional — tightened]). *The identity code distance = graph min-cut holds for topological codes (surface/toric codes) whose logical operators correspond to non-contractible homological cycles in the code graph. It does not hold for a generic overlap graph.*

Suppose the OPH observer network is equipped with a surface-code-type construction on a planar or toroidal graph G_{OPH} , and the following conditions are satisfied:

- (i) Logical X -type operators correspond to minimum-weight non-contractible cycles in the primal chain complex of G_{OPH} ; logical Z -type operators correspond to minimum-weight non-contractible cycles in the dual complex.
- (ii) Boundary conditions are such that no logical operator of weight strictly less than the min-cut of G_{OPH} exists.
- (iii) For non-planar geometries, the relevant group is $H_1(G_{\text{OPH}}; \mathbb{F}_2)$ and the code distance equals the minimum over all non-trivial homology classes of the weight of a representative cycle.

Under (i)–(iii), $d = \text{mincut}(G_{\text{OPH}})$ [12, 13].

This claim requires an explicit construction of the topological encoding map, the primal/dual complex of G_{OPH} , and verification of (i)–(iii), none of which have been provided. The claim is conditional (open issue #113).

Conjecture C.10 (Communication complexity [Conjecture]). *The OPH consensus-repair protocol, realised as a quantum communication task, has per-round complexity $O(n \cdot \text{poly}(d))$ (open issue #72; cf. [14]).*

Theorem C.11 (QECC Correspondence [Conditional / Conjecture]). *Suppose conditions (i)–(iii) of Claim C.9 hold. Then:*

- (i) [Conditional] Code distance $d = \text{mincut}(G_{\text{OPH}})$.
- (ii) [Established] The Knill–Laflamme QECC conditions [11] are satisfied for the logical subspace whenever the number of corrupted observers satisfies $t < d/2$.
- (iii) [Conjecture] Per-round communication complexity is $O(n \cdot \text{poly}(d))$.

B.4 Theorem 4 — Asynchronous Convergence

Why fairness alone does not give a probability-1 statement. Standard strong fairness guarantees that every enabled action fires infinitely often along any fair schedule; it does not impose a probability space on the set of schedules. A convergence statement of the form “converges with probability 1” requires a measure on schedules and does not follow from fairness alone. The FLP impossibility result [3] confirms that even strong fairness is insufficient for bounded-time consensus in a fully asynchronous system. The phrase “with probability 1” has been removed from Theorem C.12; the convergence is per-schedule and topological.

Additional assumptions for a quantitative bound.

- (B1) Finite known bound Δ on message delay after GST.
- (B2) Finite bound Φ on processing rates.
- (B3) $f < n/3$.

Theorem C.12 (Eventual Convergence [Established under fairness only]). *In a fully asynchronous OPH observer network under standard strong fairness, iterated application of $\mathcal{R}_{\sigma, \mathcal{N}}$ converges to a consensus state σ^* in the following sense: for every $\varepsilon > 0$ and every strongly fair schedule, there exists a step $T(\text{schedule}, \varepsilon) < \infty$ such that*

$$\frac{1}{2} \|\mathcal{R}^{\text{ot}}(\rho) - \sigma^*\|_1 < \varepsilon \quad \text{for all } t \geq T.$$

This is a per-schedule topological statement. No probability measure on schedules is assumed or needed; no uniform finite bound on T follows from fairness alone.

Theorem C.13 (Quantitative Convergence [Conditional on (B1)–(B3)]). *In a partially synchronous OPH observer network satisfying (B1)–(B3), after GST every honest observer reaches consensus within $T = O(f \cdot \Delta)$ wall-clock time (by applying the DLS framework [5], Thm. 4.4, to the OPH repair protocol; requires (B1) and (B2) explicitly and does not follow from fairness alone).*

B.5 Open Problems

- **#62** Derive repair map from OPH dynamics; verify full-rank condition for $\mathcal{N}(\sigma)$ (prerequisite for upgrading Proposition C.6 and the domain condition of Proposition C.5).
- **#63** Prove spectral gap from an OPH Lyapunov functional (upgrades Conjecture C.7 to Theorem C.8).
- **#68** Quantum observable-level confluence.
- **#69** Continuum / refinement limit of Theorems C.2 and C.13.
- **#72** Communication complexity (upgrades Conjecture C.10).
- **#73** Re-export repair map into the companion framework notation.
- **#113** Construct topological encoding map for Claim C.9.

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